WP-2025-019

Rent-seeking and Reorganisation of Administrative Units

Gopakumar Achuthankutty, Ayushi Choudhary, and Rupayan Pal

Indira Gandhi Institute of Development Research, Mumbai

July 2025

Rent-seeking and Reorganisation of Administrative Units

Gopakumar Achuthankutty, Ayushi Choudhary, and Rupayan Pal

Email(corresponding author): : ayushi@igidr.ac.in, ayushikr@gmail.com

Abstract

We examine whether rent-seeking incentives shape a central planner's decision to reorganise administrative units. In a two-stage group contest, risk-neutral administrative units compete for shares of a perfectly divisible public fund, with inter-unit and intra-unit contests occurring in Stages 1 and 2, respectively. We identify the conditions under which the planner prefers reorganisation and analyse its impact on aggregate and stage-wise rent accumulation. We show that total rent accumulation depends on the interplay between changes in fractionalisation, population inequality, and the scale effect from changes in the total population of active units following a reorganisation. While a proliferatory reorganisation (i.e., increasing the number of administrative units), when all administrative units remain active, increases the planner's rent accumulation, it can overturn the loss in social welfare under certain conditions. Furthermore, when some units become inactive, then under mild conditions on population changes, this outcome persists if the reorganisation is effectively expansive.

Keywords: Rent-seeking, Administrative Unit Reorganisation, Contest Theory, Fractionalisation, Population Inequality

JEL Code: C72, D72, H73

Rent-seeking and Reorganisation of Administrative Units*

Gopakumar Achuthankutty[1], Ayushi Choudhary[1], and Rupayan Pal[1]

[1]Indira Gandhi Institute of Development Research, Mumbai, INDIA

July, 2025

Abstract

We examine whether rent-seeking incentives shape a central planner's decision to reorganise administrative units. In a two-stage group contest, risk-neutral administrative units compete for shares of a perfectly divisible public fund, with inter-unit and intra-unit contests occurring in Stages 1 and 2, respectively. We identify the conditions under which the planner prefers reorganisation and analyse its impact on aggregate and stage-wise rent accumulation. We show that total rent accumulation depends on the interplay between changes in fractionalisation, population inequality, and the scale effect from changes in the total population of active units following a reorganisation. While a proliferatory reorganisation (i.e., increasing the number of administrative units), when all administrative units remain active, increases the planner's rent accumulation, it can overturn the loss in social welfare under certain conditions. Furthermore, when some units become inactive, then under mild conditions on population changes, this outcome persists if the reorganisation is effectively expansive.

Keywords: Rent-seeking, Administrative Unit Reorganisation, Contest Theory, Fractionalisation, Population Inequality

JEL Classification Codes: C72, D72, H73 .

*Conflicts of Interest: None

Acknowledgements: We would like to thank Kaushik Gangopadhyay, and seminar participants at IGIDR, ISI Delhi, and Winter School, Delhi School of Economics for their suggestions and comments. The usual disclaimer holds.

Corresponding Author and Address: Ayushi Choudhary, Indira Gandhi Institute of Development Research (IGIDR), Film City Road, Santosh Nagar, Goregaon (E), Mumbai 400065, India.
Emails: (Achuthankutty) gopakumar@igidr.ac.in, gopakumar.achuthankutty@gmail.com;
(Choudhary) ayushi@igidr.ac.in, ayushikr@gmail.com;
(Pal) rupayan@igidr.ac.in, rupayanpal@gmail.com

1. Introduction

Reorganisation of sub-national administrative units has gained popularity as an instrument for the implementation of decentralization reforms across the developing world. Grossman and Lewis (2014) note that around 25 sub-Saharan African countries have increased their number of administrative units by at least 20% since 1990, with Uganda's districts growing from 39 to 112 (1995–2010), Kenya's from 54 to 70 (1995–2000), and South Africa's provinces rising from 4 to 9 in 1994. Similarly, Billing (2019) documents that Burkina Faso added fifteen provinces in 1996, and Bolaji and Gariba (2020) report that Ghana increased its regions from 10 to 16 in 2018. This trend extends beyond Africa, as Indonesia and Vietnam have also witnessed significant reorganisation of their provinces (Kimura (2013); Malesky (2009)). In India, following the States Reorganisation Act (1956), total number of states has gone up from 14 to 28. In 2000, Jharkhand, Chattisgarh, and Uttarakhand were carved out of Bihar, Madhya Pradesh and Uttar Pradesh respectively, which was followed by creation of Telangana from Andhra Pradesh in 2014.

Empirical evidence on impact of administrative unit reorganisation on socio-economic outcomes presents a mixed picture. One strand of literature finds that reorganisation leads to improvements in these outcomes. For instance, Asher and Novosad (2015) document statistically significant increases in economic activity and school enrolment in India, Grossman et al. (2017) report improved public goods provision in sub-Saharan Africa, and Malesky (2009) documents that reorganisation facilitated radical economic reforms that contributed to an 8% growth rate in Vietnam. These studies attribute positive effect of reorganisation to gains in administrative efficiency in terms of improved delivery of public goods such as access to education and health, better transportation system, and streamlined infrastructure development (Grossman et al. (2017); Grossman and Lewis (2014); Kimura (2013)). Proponents of this line of argument claim that smaller governments do a better job of bringing these services to the public by promoting good governance.

On the contrary, another strand of literature documents a decline in the socio-economic indicators following a reorganisation. For instance, Vaibhav and Ramaswamy (2022) find no statistically significant improvement in per capita state GDP in India[1], Lewis (2017) observes that the infrastructure gap in the new districts in Indonesia widened by 2% over a decade compared to the parent districts, Cohen (2024) finds increased school inputs in new districts of Uganda did not translate into greater numeracy or literacy, and Billing (2019) demonstrates that fragmentation in Burkina Faso reduced public goods provision. These

[1]They argue that positive results of reorganisation on economic indicators obtained by Asher and Novosad (2015) can be attributed to Uttarakhand's extraordinary performance

adverse outcomes are frequently linked to the creation of rent-seeking opportunities, particularly on the back of infrastructure development schemes. Local demands for new administrative units often arise from deeply rooted ethno-linguistic identities and perceptions of economic marginalization (Majeed (2003)). Politicians exploit these identities to secure control over resource allocation and expand patronage networks, thereby opening avenues for rent-seeking (Lewis (2014)). In Indonesia, Lewis (2017), together with Kimura (2013) and Fitrani et al. (2005), emphasize that collusion among national and local politicians in diverting fiscal resources—often under the guise of infrastructure development—constitutes a primary motive for district splitting and the ensuing decline in service delivery.

In this context, we study how rent-seeking incentives of the central government influence it's decision to reorganise the administrative units. To this end, we consider risk-neutral administrative units that are governed by a central planner, who decides whether to reorganise these administrative units, following which it announces funds for a public project. The funds are perceived by the administrative units to be perfectly divisible. The administrative units compete for a share of this fund through a two-stage Tullock rent-seeking contest ála Katz and Tokatlidu (1996) and Stein and Rapoport (2004). In Stage 1 of this rent-seeking contest, administrative units compete among each other for the fund and in Stage 2, the economic agents (also administrative units but at a lower hierarchy) within a unit fight for the funds secured by their unit in Stage 1. Through this theoretical model, we seek answers to the following questions: (i) under what circumstances will the central government opt for reorganisation of its administrative units? (ii) What impact does reorganisation have on rent accumulation and social welfare?

Our main results are as follows. We begin by offering a complete characterization of the determinants of aggregate and stage-wise rent behaviour when administrative units undergo a reorganisation. We identify the precise condition under which the central planner opts for reorganisation, which involves a complex interplay between the changes in the number of administrative units that remain active in the contest, extent of their fractionalisation, and the scale effects of their total population. Moreover, in addition to these factors, we show that the inequality of the population distribution of active units is also crucial for determining whether the reorganisation leads to a decline in intra-unit and aggregate rent accumulation. We further expand upon these results by restricting attention to the scenario where all the administrative units remain active in the contest such that total population remains constant before and after reorganisation. A non-proliferatory reorganisation (where boundaries are redrawn without increasing the number of administrative units), increases the central planner's rent accumulation only if the society becomes more fractionalised. Furthermore, when such reorganisation is accompanied by a reduction in the population inequality, loss in social welfare cannot be alleviated as intra-group rent accumulation rises as well. Interestingly, we establish that a proliferatory reorganisation (where the number of administrative units increases) always leads to a strict increase in fractionalisation which is a significant novel insight. As a consequence, the central planner always opts for a proliferatory reorganisation. Moreover, in contrast to non-proliferatory reorganisation, if the population inequality reduces following a proliferatory reorganisation, loss in social welfare can be overturned if the proportional increase in the number of administrative units is sufficiently high.

Furthermore, we extend our analysis to examine rent accumulation in corner Nash equilibria, where not all administrative units actively engage in the rent-seeking contest. Under effectively neutral reorganisations (where the number of active units remains the same both before and after the reorganisation), we show that Stage 1, Stage 2, and total rents respond dynamically to shifts in the average population per active administrative unit, with their behaviour also influenced by changes in fractionalisation and population inequality. In this context, our analysis also considers effectively non-neutral reorganisations, encompassing both expansive (where the number of active administrative units rise after the reorganisation) and contractive (where the number of active administrative units fall after the reorganisation) scenarios. While an effectively expansive reorganisation is analogous to a proliferatory one and we find, under mild assumptions on the changes to the total population of active administrative units, leads to an increase in fractionalisation. On the other hand, also under mild assumptions on the changes to the total population of active administrative units, a significant novel insight is that an effectively contractive reorganisation consistently decreases fractionalisation. We derive precise conditions, involving thresholds on the proportional change in average active administrative unit population and specific bounds on population inequality, that determine whether Stage 1, Stage 2, or total accumulated rents strictly increase or decrease across these diverse reorganisation types. These granular findings provide a robust theoretical foundation for understanding the complex calculus of rent-seeking outcomes when participation is endogenously determined at the margins.

1.1 Related Literature

Our paper contributes to the literature on group rent-seeking and contest theory. Wärneryd (1998) finds that the unification of two regions into a centralized one leads to an increased rate of rent dissipation, however, the introduction of a federal two-stage rent-seeking structure reduces rent dissipation compared to the centralized setup. Both Münster and Staal (2012), and Hausken (2005) examine simultaneous between-group and within-group rent-seeking. Using a logistic contest success function for G equally sized groups, Münster and Staal (2012) document that in equilibrium, inter and intra-group rent-seeking do not occur simultaneously, and a larger number of groups result in lower rent dissipation. This result is similar to the one we obtain under proliferatory reorganisation when the pre-reorganisation population distribution is symmetric, or when the number of post-reorganisation units is sufficiently high. Using a one-shot rent-seeking contest for two asymmetric groups, Hausken (2005) find that free riding causes between-group fighting to go down whereas members increase within-group fighting to get a higher share of the rent. In contrast, we find that between group rent-seeking always increases under proliferatory reorganisation. Choi et al. (2016) show that in a group contest setup with two groups and within-group power asymmetries, a higher within-group rent-seeking leads to a higher between-group rent-seeking. While we do not model within-group power asymmetries in our setup, we show that, if the increase in the number groups is sufficiently low and the degree of population inequality decreases such that all administrative units participate in the contest, then reorganisation leads to higher within-group and between-group rent-seeking. Dasgupta and Neogi (2018) examine how the presence of within-group cleavages that inhibit co-ordination affect rent-seeking and welfare where two groups of equal size participate in a contest for a public good and show that an increase in fragmentation within a group reduces rent-seeking. In contrast, we establish that despite an increase in fractionalisation, total rent accumulation can decrease under certain circumstances following a proliferatory reorganisation. Bhattacharya and Rampal (2024) analyse sequential within-group and between-group contests with two asymmetric groups, each comprising two equal-sized factions, where group-level asymmetry arises from a biased inter-group contest success function. They demonstrate that compared to symmetric groups, asymmetric groups exert a higher total effort under certain cost conditions. In contrast, in our setup, asymmetry between groups arises from the population distribution of the administrative units, and we show that when the population distribution changes from symmetric to asymmetric following a proliferatory reorganisation, total effort exerted by the administrative units declines, thus reducing total rent accumulation.

The closest papers to ours in this literature are those by Katz and Tokatlidu (1996), and Stein and Rapoport (2004). Katz and Tokatlidu (1996) examine the two-stage setup discussed in this paper for two groups, and show that aggregate rent-seeking increases as inequality of the population distribution decreases. Our findings confirm that this result holds under non-proliferatory reorganisation when considering $k \geq 2$ administrative units. Stein and Rapoport (2004) extend the analysis to n groups with heterogeneous prize valuations, characterizing expressions for Stage 1 rent accumulation and expected aggregate rent accumulation for cases with equal prize valuations and equal group sizes. In our analysis, we assume uniform budget valuation across all administrative units. While it is straightforward to show that their Stage 1 rent expression is equivalent to ours, our approach uniquely expresses it in terms of fractionalisation. Moreover, we derive simplified expressions for Stage 2 rent accumulation, which reveal a negative relationship between Stage 2 rent and population distribution inequality—a feature not reported in Stein and Rapoport (2004). Finally, we extend the investigation to examine the effects of increasing the number of groups on rent accumulation at different stages, an aspect that has not been addressed in their work.

While our primary focus is on rent-seeking in the context of administrative reorganisation, our framework can also be extended to examine resource conflicts characterized by sequential inter-group and intra-group disputes, as well as to assess the impact of an increasing number of rival groups on the intensity of conflict. A substantial body of theoretical and empirical literature links fractionalisation to conflict. For instance, Collier and Hoeffler (1998) and Esteban and Ray (2008) document a non-monotonic relationship between ethnic fractionalisation and conflict—where conflict intensity initially rises with fractionalisation, peaks, and then declines—while Alesina et al. (1999) report a negative association between ethnic fractionalisation and local public goods provision. More recently, Esteban and Ray (2011) theoretically demonstrate that fractionalisation is a strong predictor of conflict when rival groups contest over a private good, a result empirically verified by Esteban et al. (2012). In contrast, our findings indicate that although fractionalisation exerts upward pressure on rent accumulation in both Stage 1 and Stage 2, an increase in population inequality can lead to a decline in total rent accumulation. Furthermore, even when inequality decreases, a sufficiently high increase in the number of administrative units following reorganisation can again result in lower total rent accumulation.

The rest of the paper is organized as follows: Section 2 presents a detailed description of the model. Section 3 presents a general comparison of stage-wise and aggregate rent accumulation. Section 4 presents the comparison with Interior Nash Equilibrium. Section 5 extends the analysis to Corner Nash Equilibrium. Section 6 offers concluding remarks. All proofs are relegated to Appendices A and B.

2. Model

We consider a rent-seeking model in a society with population N and $k(> 1)$ administrative units (referred to as AU/AUs henceforth), namely $A_1, \dots, A_k$, which are governed by a central planner (referred to as CP henceforth). An i^{th} AU consists of n_{A_i} economic agents [2], for $i = 1, \dots, k$. We assume that AUs are

[2]Here, the term economic agent is general; we can think of states of a country as AUs and districts within these states as economic agents, or districts/provinces within a state can be AUs where municipalities are the economic agents.

ordered in terms of their population sizes, i.e., $n_{A_1} \geq n_{A_2} \geq \cdots \geq n_{A_k}$.

The CP must decide whether to reorganise these k AUs to $m(\geq k)$ AUs, namely $B_1, \ldots, B_m$, with the population size of the reorganised jth unit denoted by n_{B_j}, for $j = 1, \ldots, m$ or to maintain ***status quo***, i.e., $m = k$ and $n_{B_i} = n_{A_i}$ for all $i = 1, \ldots, m(= k)$.[3] As before, we assume that AUs post-reorganisation are also ordered in terms of their population sizes, i.e., $n_{B_1} \geq n_{B_2} \geq \cdots \geq n_{B_m}$. A reorganisation is ***proliferatory*** if it leads to a strict increase in the number of AUs, i.e., $m > k$ and a reorganisation is ***non-proliferatory*** if it doesn't involve change in the number of AUs, i.e., $m = k$. Once the decision has been made, the CP announces an infrastructure/development scheme aimed at improving the delivery of public goods in the AUs for which it allocates funds of size S, normalized to 1. The AUs, in response, compete for a share of this fund in a two-stage group contest setup à la Katz and Tokatlidu (1996) as described below:

Stage 1: In the first stage of the game, the AUs compete among each other to receive the highest share of the fund by undertaking rent-seeking activities in round 1 of the group contest setup.

Stage 2: In the second stage of the game, economic agents of each AU compete within their unit for the highest share of the fund allocated to their unit at the end of the first stage.

We assume that all economic agents are homogenous. Also, rent is divisible among and within administrative units. The solution concept we use is the sub-game perfect Nash equilibrium and therefore, we solve the game by backward induction. In what follows, we describe the Tullock contests in the first and second stages of this game in detail.

2.1 Stage 2

Since we solve the game by backward induction, for the sake of convenience, we first describe the actions and payoffs of the second stage and then define the actions and payoffs of the first stage based on the optimal actions and payoffs of the second stage. In Stage 2, the pre-reorganisation individual-level allocation of the share of the fund received by each AU at the end of Stage 1 is given by

$$\theta^j_{A_i,2} = \frac{a^j_{i,2}}{\sum_{j=1}^{n_{A_i}} a^j_{i,2}}, \tag{1}$$

[3]Write that B_j's need not refer to the same AU represented by A_i's even if $i = j$ and $k = m$. Give an example to illustrate this fact.

where $\theta^j_{A_i,2}$ is the share of the fund received by each individual j in the AU A_i and $a^j_{i,2}$ is the amount of resources spent by j^{th} individual in the AU A_i for rent-seeking activities in this stage. Similarly, the post-reorganisation individual-level allocation of the share of the fund received by each AU at the end of Stage 1 is given by

$$\theta^j_{B_i,2} = \frac{b^j_{i,2}}{\sum_{j=1}^{n_{B_i}} b^j_{i,2}}, \tag{2}$$

where $\theta^j_{B_i,2}$ is the share of the fund received by each individual j in the the AU B_i and $b^j_{i,2}$ is the amount of resources spent by j^{th} individual in the AU B_i for rent-seeking activities in this stage.

The utility of an individual j in each AU A_i at this stage before reorganisation is given by

$$U^j_{A_i,2} = \theta^j_{A_i,2}\theta_{A_i,1} - a^j_{i,2} \tag{3}$$

where $\theta_{A_i,1}$ is the share of the fund received by the AU A_i in Stage 1.[4] Similarly, the utility of an individual j in each AU B_i at this stage after reorganisation is given by

$$U^j_{B_i,2} = \theta^j_{B_i,2}\theta_{B_i,1} - b^j_{i,2} \tag{4}$$

where $\theta_{B_i,1}$ is the share of the fund received by the AU B_i in Stage 1.[5]

We focus on symmetric equilibrium, i.e., $a^{j*}_{i,2} = a^*_{i,2}$ for all j in AU A_i and $b^{j*}_{i,2} = b^*_{i,2}$ for all j in AU B_i. Hence, the second stage optimal payoffs are given by $U^{j*}_{A_i,2} = U^*_{A_i,2}$ and $U^{j*}_{B_i,2} = U^*_{B_i,2}$.

Lastly, we define the total rent generated in this stage before reorganisation as $R^{Pre}_2 = \sum_{i=1}^{k} n_{A_i} a^*_{i,2}$ and the total rent generated in this stage after reorganisation as $R^{Post}_2 = \sum_{i=1}^{m} n_{B_i} b^*_{i,2}$.

2.2 Stage 1

In this stage, the AUs compete with each other for the highest share of the fund. We follow Tullock (1980) in specifying an AU's share of the fund which is determined by aggregate rent seeking by its agents relative to aggregate rent seeking by agents of all the other AUs. The allocation of the fund

[4]The term $\theta_{A_i,1}$ will be defined formally in the next section.
[5]The term $\theta_{B_i,1}$ will be defined formally in the next section.

pre-reorganisation for each AU is given by

$$\theta_{A_i,1} = \frac{\sum_{j=1}^{n_{A_i}} a_{i,1}^j}{\sum_{i=1}^{k}\sum_{j=1}^{n_{A_i}} a_{i,1}^j}, \tag{5}$$

where $\theta_{A_i,1}$ is the share of fund received by the AU A_i and $a_{i,1}^j$ is the amount of resources spent by j^{th} individual in the AU A_i for rent-seeking activities in this stage. Similarly, the allocation of the fund post reorganisation for each AU is given by

$$\theta_{B_i,1} = \frac{\sum_{j=1}^{n_{B_i}} b_{i,1}^j}{\sum_{i=1}^{m}\sum_{j=1}^{n_{B_i}} b_{i,1}^j}, \tag{6}$$

where $\theta_{B_i,1}$ is the share of fund received by the AU B_i and $b_{i,1}^j$ is the amount of resources spent by j^{th} individual in the AU B_i for rent-seeking activities in this stage.

The utility of an individual j in AU A_i at this stage before reorganisation is given by

$$U_{A_i,1}^j = U_{A_i,2}^* - a_{i,1}^j \tag{7}$$

and utility of an individual j each AU B_i at this stage after reorganisation is given by

$$U_{B_i,1}^j = U_{B_i,2}^* - b_{i,1}^j \tag{8}$$

Since we focus on symmetric equilibrium, the optimal action of each individual in the AU A_i is $a_{i,1}^*$ where $i = 1, \ldots, k$ and the optimal action of each individual in the AU B_i is $b_{i,1}^*$ where $i = 1, \ldots, m$. Similarly, the optimal payoff of the AU A_i is $U_{A_i,1}^*$ where $i = 1, \ldots, k$ and the optimal payoff of the AU B_i is $U_{B_i,1}^*$ where $i = 1, \ldots, m$. Lastly, we define the total rent generated in this stage pre-reorganisation as $R_1^{Pre} = \sum_{i=1}^{k} n_{A_i} a_{i,1}^*$ and the total rent generated in this stage after reorganisation as $R_1^{Post} = \sum_{i=1}^{m} n_{B_i} b_{i,1}^*$.

We start with four preliminary results. Our first result establishes that, in general, not all AUs exert positive effort in equilibrium.

Fact 1. *In the rent-seeking contest before reorganisation, at equilibrium, there exists an AU with index i_k^*, with $1 \leq i_k^* \leq k-1$, such that all agents in AUs with index $i \geq i_k^*$ exert positive (symmetric) effort and all agents in AU with index $i < i_k^*$ exert zero effort. Similarly, in the rent-seeking contest after reorganisation, at equilibrium, there exists an AU with index i_m^*, with $1 \leq i_m^* \leq m-1$, such that all agents in AUs with index $i \geq i_m^*$ exert positive (symmetric) effort and all agents in AU with index $i < i_m^*$ exert zero effort.*

Fact 1 tells us that prior to reorganisation, only AUs with $i \geq i_k^*$, and hence, we notate the number of active AUs before reorganisation by $\hat{k} = k - i_k^* + 1$. Similarly, after reorganisation, only AUs with $i \geq i_m^*$ remain active, implying that the total number of active AUs becomes $\hat{m} = m - i_m^* + 1$. From Fact 1, we know that $1 \leq i_k^* \leq k-1$ and $1 \leq i_m^* \leq m-1$, and therefore, $\hat{m}, \hat{k} > 1$. Therefore, the total population of $\hat{k}$ AUs before reorganisation is $N_{\hat{k}}^{Pre} = \sum_{i=i_k^*}^{k} n_{A_i}$ before reorganisation and the total population of $\hat{m}$ AUs after reorganisation is $N_{\hat{m}}^{Post} = \sum_{i=i_m^*}^{m} n_{B_i}$.

The comparative statics of rent accumulation across the two stages of the rent-seeking contest before and after the reorganisation crucially depend on the relative symmetry or asymmetry of the population distribution. Since, by Fact 1, not all AUs are active in equilibrium, we only need to consider the population distribution of the active AUs. Consequently, unless explicitly stated, a population distribution will henceforth refer to the population distribution of active AUs, which we find convenient to express as population shares. That is, let $s_{A_i} = \frac{n_{A_i}}{N_{\hat{k}}^{Pre}}$ for all $i_k^* \leq i \leq k$ and let $s_{B_i} = \frac{n_{B_i}}{N_{\hat{m}}^{Post}}$ for $i_m^* \leq i \leq m$. When the population distribution is symmetric both before and after reorganisation then $s_{A_i} = \frac{1}{\hat{k}}$ for $i_k^* \leq i \leq k$ and $s_{B_i} = \frac{1}{\hat{m}}$ for $i_m^* \leq i \leq m$. We consider an asymmetric population distribution as a perturbation from a symmetric population distribution. When the population distribution is asymmetric before redistribution, we have $s_{A_i} = \frac{1}{\hat{k}} + \Delta_{A_i}$ for $i_k^* \leq i \leq k$ where $\Delta_{A_{i_k^*}} \geq \Delta_{A_{i_k^*+1}} \geq \cdots \geq \Delta_{A_k}$ such that $\sum_{i=i_k^*}^{k} \Delta_{A_i} = 0$. Similarly, when the population distribution is asymmetric after redistribution, we have $s_{B_i} = \frac{1}{\hat{m}} + \Delta_{B_i}$, where $\Delta_{B_{i_m^*}} \geq \Delta_{B_{i_m^*+1}} \geq \cdots \geq \Delta_{B_{\hat{m}}}$ such that $\sum_{i=i_m^*}^{m} \Delta_{B_i} = 0$. It is important to note that both Δ_{A_i}'s and Δ_{B_i}'s for an active AU before and after reorganisation can be positive as well as negative numbers.

The next result derives the equilibrium rent expression in Stages 1 and 2 of the rent-seeking contest before and after reorganisation.

Fact 2. *Given the optimal actions and payoffs of the economic agents, in equilibrium:*

(i) The expression for the total rent extracted pre-reorganisation in Stage 1 is

$$R_1^{Pre} = \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} n_{A_i}^2} = \frac{(k - i_k^*)}{\left(N_{\hat{k}}^{Pre}\right)^2 \sum_{i=i_k^*}^{k} s_{A_i}^2} \tag{9}$$

where i_k^* is the smallest index of an active AU pre-reorganisation and post-reorganisation is

$$R_1^{Post} = \frac{(m - i_m^*)}{\sum_{i=i_m^*}^{m} n_{B_i}^2} = \frac{(m - i_m^*)}{\left(N_{\hat{m}}^{Post}\right)^2 \sum_{i=i_m^*}^{m} s_{B_i}^2} \tag{10}$$

where i_m^* is the smallest index of an active AU post-reorganisation; and

(ii) The expression for the total rent extracted pre-reorganisation in Stage 2 is

$$R_2^{Pre} = \left[1 + \frac{(k - i_k^*)\sum_{i=i_k^*}^{k} n_{A_i}}{\sum_{i=i_k^*}^{k} n_{A_i}^2} - \sum_{i=i_k^*}^{k} \frac{1}{n_{A_i}}\right] = \frac{1}{N_{\hat{k}}^{Pre}}\left[N_{\hat{k}}^{Pre} + \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}\right] \tag{11}$$

and post-reorganisation is

$$R_2^{Post} = \left[1 + \frac{(m - i_m^*)\sum_{i=i_m^*}^{m} n_{B_i}}{\sum_{i=i_m^*}^{m} n_{B_i}^2} - \sum_{i=i_m^*}^{m} \frac{1}{n_{B_i}}\right] = \frac{1}{N_{\hat{m}}^{Post}}\left[N_{\hat{m}}^{Post} + \frac{(m - i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}\right] \tag{12}$$

The next result specifies the conditions for an AU to remain active in Stages 1 and 2 of the game both before and after reorganisation.

Fact 3. *Assume that, in equilibrium, for every AU A_i with $i \geq i_k^*$ and every AU B_i with $i \geq i_m^*$, economic agents choose positive effort for rent-seeking in all Stages of the game. Then, in Stage 1, the following conditions must hold: for the AU A_i, $\sum_{h=i_k^*}^{k} n_{A_h}^2 - (k - i_k^*)\, n_{A_i}^2 = \sum_{h=i_k^*}^{k} n_{A_h}^2 - (\hat{k} - 1)\, n_{A_i}^2 > 0$, and for the AU B_i, $\sum_{h=i_m^*}^{m} n_{B_h}^2 - (m - i_m^*)\, n_{B_i}^2 = \sum_{h=i_m^*}^{m} n_{B_h}^2 - (\hat{m} - 1)\, n_{B_i}^2 > 0$.*

We henceforth refer to the above conditions as ***General Nash Equilibrium (GNE) conditions***.

Lastly, we establish a general result identifying the population distribution that maximizes the sum of squares of active AU populations when a total population of size N is divided into k AUs. In this context, a population distribution is a collection $\left(n_1, n_2, \ldots, n_{i_k^*}, \ldots, n_k\right)$ where $n_1 \geq n_2 \geq \cdots \geq n_{i_k^*} \geq \cdots \geq n_k$ and all AUs with $i_k^* \leq i < k$ remain active in the contest such that $\sum_{i=i_k^*}^{k} n_i = N_{\hat{k}}$. The population distribution of the active AUs is given by the collection $\left(n_{i_k^*}, n_{i_k^*+1}, \ldots, n_k\right)$.

Fact 4. *Fix a total population size N that is divided into k AUs such that all AUs with $i_k^* \leq i < k$ remain active in the contest, the number of active AUs is given by $\hat{k} = k - i_k^* + 1$, the total population of active AUs is given by $\sum_{i=i_k^*}^{k} n_i = N_{\hat{k}}$, and $N_{\hat{k}} > (\hat{k}-1)\hat{k}$. Then, the population distribution $(n_{i_k^*}, n_{i_k^*+1}, \ldots, n_k)$ maximizes $\sum_{i=i_k^*}^{k} n_i^2$ subject to the constraints $\sum_{j=i_k^*}^{k} n_j^2 > (k - i_k^*)n_i^2$, for all $i = i_k^*, \ldots, k$ if and only if*

$$n_i = \left(\frac{N_{\hat{k}} - r_{\hat{k}}}{\hat{k}-1}\right) \quad \textit{for } i = i_k^*, \ldots, k-1, \quad \textit{and} \quad n_k = r_{\hat{k}},$$

where $r_{\hat{k}}$ is the smallest number so that $(\hat{k}-1)\left(\frac{N_{\hat{k}} - r_{\hat{k}}}{\hat{k}-1}\right) + r_{\hat{k}} = N_{\hat{k}}$. When a population of size N is divided among m AUs, the corresponding conditions are obtained by replacing $\hat{k}$ with $\hat{m}$ in the above.

The proof of this fact is in Appendix B. The intuition behind this result stems from the nature of maximizing the sum of squares. Without the constraints, the sum of squares of active AUs is maximized by the most asymmetric distribution, i.e., $(N_{\hat{k}} - \hat{k} + 1, 1, \ldots, 1)$. Since $n_{i_k^*} \geq \cdots \geq n_k$, observe that the constraint in the maximization exercise is equivalent to $\frac{1}{(\hat{k}-1)} \sum_{j=i_k^*}^{k} n_j^2 > n_{i_k^*}^2$. Note that this constraint prevents extreme asymmetry and implies a certain level of "evenness" among the population sizes. Specifically, this constraint necessitates that the first $(\hat{k}-1)$ population sizes become equal, i.e., $n_{i_k^*} = n_{i_k^*+1} = \cdots = n_{k-1}$, and the maximization process then tries to maximize the difference between this shared value and the last population size, n_k.

3. Rent Comparison: General Results

Our main results compare the rent accumulated in all stages of the rent-seeking contest and the total rent accumulated before and after reorganisation when the reorganisation is non-proliferatory and proliferatory. Before moving on to these results, we first derive equivalent conditions (i) when the rent accumulated in Stage 1 is higher after the reorganisation, (ii) when the rent accumulated in Stage 2 is lower after the reorganisation, and (iii) when the total rent accumulated is lower after the reorganisation.

Note that the CP's utility from any reorganisation is based on the magnitude of rent accumulated in

Stage 1 of the rent-seeking contest. That is, the CP weakly (or strictly) prefers a reorganisation to the status quo if $R_1^{Post} \geq R_1^{Pre}$ (or $R_1^{Post} > R_1^{Pre}$) and weakly (or strictly) prefers status quo to a reorganisation if $R_1^{Pre} \geq R_1^{Post}$ (or $R_1^{Pre} > R_1^{Post}$).

Theorem 3.1. *Any reorganisation weakly increases the rent accumulation in Stage 1 of the rent-seeking contest if and only if*

$$\frac{\hat{m}\omega_{\hat{m}}^{Post}}{\hat{k}\omega_{\hat{k}}^{Pre}} \geq \left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2,$$

where $\omega_{\hat{k}}^{Pre} = \dfrac{(\hat{k}-1)}{\hat{k}\sum\limits_{i=i_k^*}^{k} s_{A_i}^2}$, *and* $\omega_{\hat{m}}^{Post} = \dfrac{(\hat{m}-1)}{\hat{m}\sum\limits_{i=i_m^*}^{m} s_{B_i}^2}$.

Before providing an intuitive explanation of Theorem 3.1, note that $\omega_{\hat{k}}^{Pre}$ and $\omega_{\hat{m}}^{Post}$ are increasing functions of the degree of fractionalisation of the population distribution of active AUs before and after the reorganisation respectively.[6] Therefore, Theorem 3.1 states that any reorganisation weakly increases the rent accumulated in Stage 1 if and only if the combined effect of the change in the number of active AUs and the degree of fractionalisation of the active AU population distribution, represented by the LHS of the inequality, outweighs the scale effect due to the change in the total population of active AUs, represented by the RHS of the inequality. On the contrary, the theorem says that any reorganisation weakly decreases the rent accumulated in Stage 1 if and only if the combined effect of the change in the number of active AUs and the degree of fractionalisation of the active AU population distribution is contained by the scale effect due to the change in the total population of active AUs.

Before we move on to our next theorem comparing Stage 2 rents, observe that an equivalent way of writing the Stage 2 rent expressions are as follows:

$$R_2^{Pre} = 1 + \frac{1}{N_{\hat{k}}^{Pre}} R_1^{Pre} - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}$$
$$R_2^{Post} = 1 + \frac{1}{N_{\hat{m}}^{Post}} R_1^{Post} - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}$$

[6] The degree of fractionalisation within a society is commonly measured using the Hirschman-Herfindahl index of fractionalisation (Esteban and Ray (2011); Garcia Montalvo and Reynal-Querol (2002)). In our setting, the fractionalisation index before the reorganisation is given by $1 - \sum\limits_{i=i_k^*}^{k} s_{A_i}^2$ and the fractionalisation index after the reorganisation is given by $1 - \sum\limits_{i=i_m^*}^{m} s_{B_i}^2$. Intuitively, the fractionalisation index represents the probability that two randomly selected economic agents belong to different AUs.

Clearly, the rent accumulated in the Stage 2 is a function of rent accumulated in Stage 1 of the rent-seeking contest. This is not surprising given the cumulative nature of the rent-seeking contest - in the second stage, the economic agents of each AU are competing for the share of the funds it received in the first stage of the contest. This means that the rent accumulated in Stage 2 before and after the reorganisation increases with an increase in fractionalisation of the pre-reorganisation and post-reorganisation population distributions respectively.

We denote the last term in the Stage 2 rent expressions as $\tau_{\hat{k}}^{Pre}$ and $\tau_{\hat{m}}^{Post}$, where $\tau_{\hat{k}}^{Pre} = \frac{1}{\hat{k}} \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} = \frac{\hat{k}}{H_{\hat{k}}^{Pre}}$ and $\tau_{\hat{m}}^{Post} = \frac{1}{\hat{m}} \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} = \frac{\hat{m}}{H_{\hat{m}}^{Post}}$.[7] Here, $H_{\hat{k}}^{Pre}$ represents the ratio of the harmonic to the arithmetic mean of the population shares of active AUs before the reorganisation, and $H_{\hat{m}}^{Post}$ represents the same ratio after the reorganisation. Consequently, $\tau_{\hat{k}}^{Pre}$ and $\tau_{\hat{m}}^{Post}$ are functions of the number of active AUs and the ratio of harmonic to arithmetic mean of their population shares. The latter ratio is related to another measure of the symmetry of the population distribution based on the Atkinson's inequality measure where the inequality aversion parameter is 2 when expressed in terms of the population shares of active AUs.[8] To justify its use as a measure of asymmetry of the population distribution of active AUs, the equally distributed equivalent population size is the harmonic mean of the population distribution which is always lesser than or equal to the arithmetic mean of the population distribution and these values are equal if and only if the population distribution is symmetric. Therefore, the Atkinson's measure of population inequality (with parameter 2) is always less than or equal to 1 and it achieves its maximum only when the population distribution is symmetric. In order to distinguish this measure of asymmetry from the degree of fractionalisation, we call this as measure of population inequality.[9] Therefore, $\tau_{\hat{k}}^{Pre}$ and $\tau_{\hat{m}}^{Post}$ are increasing functions of the degree of inequality in the respective population distributions. Moreover, the τ value of a population distribution increases with the the number of active AUs.

---

[7]See Appendix A where we establish that $\sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} = \frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}$ and $\sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} = \frac{\hat{m}^2}{H_{\hat{m}}^{Post}}$.

[8]In the context of income inequality, the Atkinson's inequality index (Atkinson (1970), Yalonetzky (2020)) is defined as

$$A_\varepsilon = 1 - \frac{y_\varepsilon^*}{\overline{y}}$$

where y_ε^* is the equally distributed equivalent income when the inequality aversion parameter is ε and $\overline{y}$ is the arithmetic mean income. The Atkinson's index is based on the Constant Relative Risk Aversion (CRRA) utility function with the risk aversion parameter being ε. One can see that when $\varepsilon = 2$ then y_ε^* is simply the harmonic mean of the income distribution.

[9]To the best of our knowledge, these are independent measures and there is no known mathematical relationship between the two.

Theorem 3.2. *Any reorganisation weakly decreases the rent accumulated in Stage 2 if and only if*

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \omega_{\hat{k}}^{Pre}\right)} \geq \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}.$$

First note that the LHS in the inequality in Theorem 3.2 captures the change in the combined effect of the number of active AUs and the difference between the τ and ω values as a result of the reorganisation. From the discussion in the paragraph before the statement of this theorem and the discussion on Theorem 3.1, we can conclude that the expression capturing the difference in τ and ω values is an increasing function of the level of population inequality and a decreasing function of the degree of fractionalisation of the population distribution of active AUs. As before, the theorem says that any reorganisation weakly decreases the rent accumulated in Stage 2 of the rent-seeking contest if and only if the change in the combined effect of the number of active AUs and the difference in τ and ω values is higher than the scale effect due to the change in the total population of active AUs as a result of the reorganisation. On the contrary, any reorganisation weakly increases the rent accumulated in Stage 2 of the rent-seeking contest if and only if the scale effect due to the change in the total population of active AUs as a result of the reorganisation contains the change in the combined effect of the number of active AUs and the difference in τ and ω values.

Next, we derive the expression for the total rent accumulated in the rent-seeking contest before and after the reorganisation. Since $R^{Pre} = R_1^{Pre} + R_2^{Pre}$, we have

$$R^{Pre} = \frac{1}{N_{\hat{k}}^{Pre}}\left[N_{\hat{k}}^{Pre} + \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2}\left(\frac{N_{\hat{k}}^{Pre} + 1}{N_{\hat{k}}^{Pre}}\right) - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}\right] \tag{13}$$

and since $R^{Post} = R_1^{Post} + R_2^{Post}$, we have

$$R^{Post} = \frac{1}{N_{\hat{m}}^{Post}}\left[N_{\hat{m}}^{Post} + \frac{(m - i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2}\left(\frac{N_{\hat{m}}^{Post} + 1}{N_{\hat{m}}^{Post}}\right) - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}\right]. \tag{14}$$

The following theorem establishes how the comparison of total rent accumulated in the rent-seeking contest is connected to the level of inequality and fractionalisation in the population distribution.

Theorem 3.3. *Any reorganisation weakly decreases the total rent accumulated if and only if*

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\omega_{\hat{k}}^{Pre}\right)} \geq \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}.$$

While the interpretation of Theorem 3.3 largely parallels that of Theorem 3.2 given that total rent is the sum of Stage 1 and Stage 2 rents, notable differences in their respective inequalities exist. Firstly, the expression for the difference between τ and ω values in Theorem 3.3 assigns a higher weight (greater than 1) to ω. This amplified weightage, a direct consequence of the cumulative nature of the rent-seeking contest, is a function of the total population size of active AUs. Secondly, this dependence of the LHS expression on the total population size of active AUs inherently prevents us from isolating the scale effect arising from changes in the total population to act as a threshold on the RHS of the inequality.

Moving forward, we'll examine how any reorganisation of AUs affects rent accumulation in the rent-seeking contest, specifically focusing on the nature of the population distribution of active AUs. Section 4 begins by analysing the scenario where all economic agents exert positive effort. This assumption yields particularly definitive and insightful results regarding the behaviour of rent accumulation. Following this, Section 5 broadens our analysis to situations where not all agents exert positive effort in equilibrium. We observe that the core intuition established in Section 4 largely holds, provided appropriate assumptions are made concerning the scale effects tied to the population sizes of active AUs.

4. Rent Comparison with Interior Nash Equilibrium

In this section, we restrict our analysis to the scenarios where all economic agents exert positive effort in equilibrium in all stages of the game, i.e., we restrict our attention to only interior Nash equilibrium (INE) in all stages of the rent-seeking contest. This means that the smallest index of the active group before and after reorganisation are $i_k^* = 1$ and $i_m^* = 1$ respectively and the size of the total population of active AUs before and after reorganisation is N. We assume that the total population size, N, is large enough with respect to the number of AUs after reorganisation, m, and therefore, in what follows, we assume that $N > m(m-1)$ so as to guarantee the existence of a population distribution that admits an INE. Lastly, the population share of the AU i before reorganisation is $s_{A_i} = \frac{n_{A_i}}{N}$ and after reorganisation is $s_{B_i} = \frac{n_{B_i}}{N}$.

In this context, if a population distribution is symmetric both before and after reorganisation then

$s_{A_i} = \dfrac{1}{k}$ for all $i = 1, \ldots, k$ and $s_{B_i} = \dfrac{1}{m}$ for all $i = 1, \ldots, m$. As before, we view an asymmetric population distribution as a perturbation from a symmetric population distribution. In other words, if the population distribution is asymmetric before reorganisation then it is convenient to write $s_{A_i} = \dfrac{1}{k} + \Delta_{A_i}$ such that $\Delta_{A_1} \geq \Delta_{A_2} \geq \cdots \geq \Delta_{A_k}$ and $\sum_{i=1}^{k} \Delta_{A_i} = 0$. Similarly, if the the population distribution is asymmetric before reorganisation then we write $s_{B_i} = \dfrac{1}{m} + \Delta_{B_i}$ such that $\Delta_{B_1} \geq \Delta_{B_2} \geq \cdots \geq \Delta_{B_m}$ and $\sum_{i=1}^{m} \Delta_{B_i} = 0$.

Setting $i_k^* = i_m^* = 1$, $\hat{k} = k$, $\hat{m} = m$, and $N_{\hat{k}}^{Pre} = N_{\hat{m}}^{Post} = N$, we derive the equivalent conditions for the comparison of rent accumulated in Stages 1 and 2 and comparison of the total rent accumulated from Theorems 3.1-3.3, which are collected in Theorem 4.1.

Theorem 4.1. *When all AUs participate in the rent-seeking contest, the following statements hold:*

(i) Any reorganisation weakly increases (weakly decreases) the rent accumulated in Stage 1 of the rent-seeking contest if and only if $m\omega_m^{Post} \geq (\leq) k\omega_k^{Pre}$.

(ii) Any reorganisation weakly decreases (weakly increases) the rent accumulated in Stage 2 of the rent-seeking contest if and only if $m\left(\tau_m^{Post} - \omega_m^{Post}\right) \geq (\leq) k\left(\tau_k^{Pre} - \omega_k^{Pre}\right)$.

(iii) Any reorganisation weakly increases (weakly decreases) the total rent accumulated in the rent-seeking contest if and only if $m\left(\tau_m^{Post} - \left(\dfrac{N+1}{N}\right)\omega_m^{Post}\right) \geq (\leq) k\left(\tau_k^{Pre} - \left(\dfrac{N+1}{N}\right)\omega_k^{Pre}\right)$.

It is evident from Theorem 4.1 that we are able to neutralize the scale effects due to changes in the total population of active AUs when restricting attention to interior Nash equilibrium.

4.1 Non-proliferatory Reorganisation

In this section, we compare the rent accumulated in the rent-seeking contest in the case of a non-proliferatory reorganisation, i.e., when $m = k$. Given this additional assumption, changes in rent accumulation in both stages of the rent-seeking contest will now solely depend on the degree of fractionalisation and the level of population inequality.

Theorem 4.2. *When comparing the rent accumulated in Stage 1, we obtain the following results:*

(i) Any reorganisation will not change the rent accumulated in which the population distribution remains symmetric before and after the change.

(ii) Any reorganisation will strictly decrease the accumulated rent in which the population distribution is symmetric before the change and asymmetric after it.

(iii) Any reorganisation will strictly increase the accumulated rent in which the population distribution is asymmetric before the change and symmetric after it.

Since the number of AUs (k and m) and total population (N) doesn't affect rent accumulation, the intuition behind the result is rather straightforward: any reorganisation increases the rent accumulation if and only if it makes the population distribution more fractionalised. Note that the above result is silent about the case when the population distribution is asymmetric both before and after the reorganisation. In this case, we can say that the rent accumulated in Stage 1 weakly increases if and only if $\sum_{i=1}^{k} \Delta_{B_i}^2 \leq \sum_{i=1}^{k} \Delta_{A_i}^2$.

Theorem 4.3. *When comparing the rent accumulated in Stage 2, we obtain the following results:*

(i) Any reorganisation will not change the rent accumulated in which the population distribution remains symmetric before and after the change.

(ii) Any reorganisation will strictly decrease the accumulated rent in which the population distribution is symmetric before the change and asymmetric after it.

(iii) Any reorganisation will strictly increase the accumulated rent in which the population distribution is asymmetric before the change and symmetric after it.

The underlying intuition behind Theorem 4.3 is that the change in rent accumulated in Stage 2 of the rent-seeking contest is completely determined by how the reorganisation changes the degree of fractionalisation, and the changes in the inequality of the population distribution. In the case where the population distribution is asymmetric before and after the reorganisation, about which the theorem is silent, any reorganisation weakly decreases (weakly increases) the rent accumulated in Stage 2 of the rent-seeking contest if and only if $\sum_{i=1}^{k} \Delta_{B_i}^2 \geq (\leq) \sum_{i=1}^{k} \Delta_{A_i}^2$ and $H_k^{Pre} \geq (\leq) H_k^{Post}$.

Theorem 4.4. *When comparing the total rent accumulated, we obtain the following results:*

(i) Any reorganisation will not change the rent accumulated in which the population distribution remains symmetric before and after the change.

(ii) Any reorganisation will strictly increase the accumulated rent in which the population distribution is asymmetric before the change and symmetric after it.

(iii) *Any reorganisation will strictly decrease the accumulated rent in which the population distribution is symmetric before the change and asymmetric after it.*

As in the case of other results in this section, the above theorem is silent about the case when the population distribution is asymmetric both before and after the change. In this case, we say that a non-proliferatory reorganisation weakly decreases the total rent accumulated if and only if $\sum_{i=1}^{k} \Delta_{B_i}^2 \geq \sum_{i=1}^{k} \Delta_{A_i}^2$ and $H_k^{Pre} \geq H_k^{Post}$.

From Theorem 4.3 and Theorem 4.4 observe that for total rent and Stage 2 rent to increase (i) fractionalisation must rise, and (ii) degree of population inequality must reduce. This happens because as fractionalisation increases after reorganisation, it unambiguously increases Stage 1 rent. Additionally, a reduction in inequality combined with an increase in fractionalisation increases Stage 2 rent as well. Consequently, total rent also goes up in this scenario.

4.2 Proliferatory Reorganisation

In this section, we compare the rent accumulated in the rent-seeking contest in the case of proliferatory reorganisation, i.e., when $m > k$. Our first result here examines the behaviour of rent accumulation in Stage 1 under a proliferatory reorganisation.

Theorem 4.5. *Any reorganisation strictly increases the rent accumulation in Stage 1.*

The intuition behind the above theorem is that any proliferatory reorganisation increases the fractionalisation in the population distribution of AUs. First, consider the scenario where the post-reorganisation population distribution is symmetric. Since $m > k$, in this case, we have $\sum_{i=1}^{k} s_{A_i}^2 = \frac{1}{k} + \sum_{i=1}^{k} \Delta_{A_i}^2 > \frac{1}{m} + \sum_{i=1}^{k} \Delta_{A_i}^2 \geq \frac{1}{m} = \sum_{i=1}^{m} s_{B_i}^2$, thereby implying an increase in fractionalisation after the reorganisation.

Next, consider the scenario where the pre-reorganisation population distribution is symmetric and the post-reorganisation population distribution is asymmetric. Given a symmetric pre-reorganisation population distribution, each AU has a population of $n_{A_i} = \frac{N}{k}$ for all $i = 1, \ldots, k$.[10] From Fact 4, among all post-reorganisation population distributions permitting an interior Nash equilibrium when the total population size is N and the number of (active) AUs is m, the one that maximizes the sum of squares is $n_{B_i} = \left(\frac{N - r_m}{m - 1}\right)$ for $i = 1, \ldots, m - 1$, and $n_{B_m} = r_m$, where $1 \leq r_m \leq m - 1$ is the smallest integer

[10]Here, we assume that the pair N and k admits a symmetric distribution, i.e., k perfectly divides N.

satisfying $(m-1)\left(\frac{N-r_m}{m-1}\right) + r_m = N$ and $r_m < \left(\frac{N-r_m}{m-1}\right)$. Since $r_m \leq m-1 < N$ and $k \leq m-1$, we obtain

$$\sum_{i=1}^{m} n_{B_i}^2 = (m-1)n_{B_1}^2 + n_{B_m}^2 = \frac{(N-r_m)^2 + (m-1)r_m^2}{m-1} = \frac{N^2 + r_m(mr_m - 2N)}{m-1} < \frac{N^2}{k} = \sum_{i=1}^{k} n_{A_i}^2,$$

thereby implying an increase in fractionalisation.

Finally, consider the scenario where the population distribution is asymmetric both before and after the reorganisation. Here, observe that any asymmetric pre-reorganisation population distribution is more fractionalised than a symmetric pre-reorganisation distribution. Combining this observation with the above discussion, we find that fractionalisation increases yet again, leading to an increase in Stage 1 rent accumulation. Hence, irrespective of population asymmetry, the CP always opts for proliferatory reorganisation.

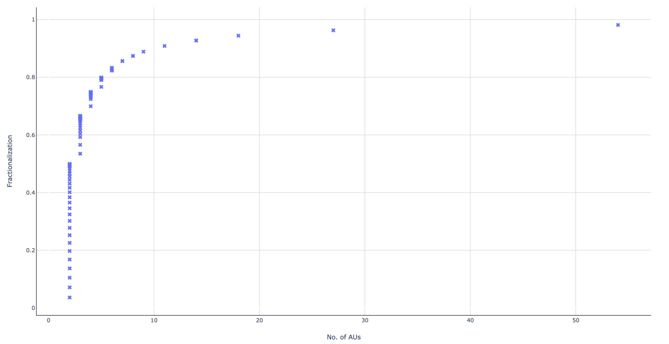

Figure 1: Relationship between Proliferation and Fractionalisation ($N = 54$)

We use Figure 1 to further illustrate how proliferatory reorganisation increases fractionalisation. In the figure, the total population is fixed at $N = 54$. The figure plots the fractionalisation levels of all population distributions that admit an interior Nash equilibrium. As is evident from the figure, the highest level of fractionalisation when there are two AUs is below the lowest level of fractionalisation when there are three AUs and this trend continues as we increase the number of AUs.

Theorem 4.6. *When comparing the rent accumulated in Stage 2, we obtain the following results:*

(i) Any reorganisation weakly decreases the accumulated rent in which the population distribution is symmetric before the change.

(ii) When the population distribution is asymmetric before the reorganisation then:

(a) any reorganisation weakly decreases the accumulated rent if and only if

$$\frac{(m-k)}{k} \geq k\left(\frac{1}{H_k^{Pre}} - \frac{1+\sum_{i=1}^{k}\Delta_{A_i}^2}{1+k\sum_{i=1}^{k}\Delta_{A_i}^2}\right),$$

when the population distribution is symmetric after the reorganisation; and

(b) any reorganisation strictly decreases the accumulated rent if

$$\frac{(m-k)}{k} \geq k\left(\frac{1}{H_k^{Pre}} - \frac{1+\sum_{i=1}^{k}\Delta_{A_i}^2}{1+k\sum_{i=1}^{k}\Delta_{A_i}^2}\right),$$

when the population distribution is asymmetric after the reorganisation.

The intuition for Theorem 4.6 (i) is simple: if the population distribution is symmetric before a reorganisation, fractionalisation will strictly increase while population inequality will at least weakly increase, irrespective of whether the post-reorganisation distribution remains symmetric or becomes asymmetric. Hence, the rent-decreasing effect of higher inequality dominates the rent-increasing effect of higher fractionalisation, thereby reducing rent accumulation in Stage 2 following a reorganisation.

When the population distribution is asymmetric before the reorganisation, Theorem 4.6 (ii) provides a condition, expressed as a threshold for the proportional increase in the number of AUs, that dictates the behaviour of rent accumulation. Specifically, this condition is necessary and sufficient for the accumulated rent to weakly decrease if the post-reorganisation distribution is symmetric, and sufficient for it to strictly decrease if the post-reorganisation distribution is asymmetric. It is evident from the the threshold expression,

$$k\left(\frac{1}{H_k^{Pre}} - \frac{1+\sum_{i=1}^{k}\Delta_{A_i}^2}{1+k\sum_{i=1}^{k}\Delta_{A_i}^2}\right),$$

that the threshold increases with an increase in population inequality and decreases with an increase in fractionalisation.

To see why the threshold condition for the proportional increase in AUs is relevant, let us first consider a reorganisation that changes an asymmetric population distribution into a symmetric one. In this case, Lemma B.3[11] implies that fractionalisation again strictly increases, but the degree of inequality of the distribution decreases (since $H_k < H_m$). Therefore, for the Stage 2 rents to weakly decline, the proportional increase in the number of AUs must be high enough to compensate for this drop. For example, with $N = 54$

[11]See Appendix B.

and $k = 4$, let the population distribution before the reorganisation be $(16, 15, 15, 8)$. The threshold value is approximately 0.5, implying that there must be at least a 50% proportional increase in AUs or in other words, m must at least be 6. Instead, if the population distribution before reorganisation is $(17, 17, 17, 3)$, where fractionalisation is lower and inequality is higher, the threshold rises to approximately 3.4, requiring m to be at least 18 for rent accumulation in Stage 2 to decline.

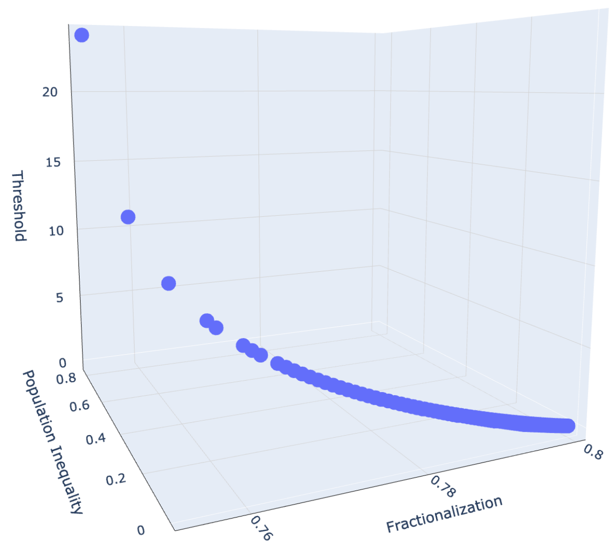

Figure 2: Relationship between Fractionalisation, Population Inequality, and the Threshold when $N = 500$ and $k = 5$

Figure 2 depicts the dynamics of how the threshold changes with a change in population inequality and fractionalisation (of the pre-reorganisation population distribution) when the total population size 500 and the number of AUs is 5. The figure clearly illustrates our previous observation: the threshold rises with greater population inequality and falls with increased fractionalisation. Furthermore, the figure exhibits a slight tilt towards the axis representing fractionalisation, thereby suggesting that the threshold exhibits higher sensitivity to changes in fractionalisation compared to changes in population inequality. For example, note that when reducing the threshold from 23.97 to 11.45 requires only a 0.0038 increase in fractionalisation (from 0.754), whereas it requires a 0.14 decrease in population inequality (from 0.82). Similarly, the threshold drops from 3.03 to 1.95 when fractionalisation increases from 0.771 to 0.777, while population inequality decreases from 0.336 to 0.236. This inclination underscores that the threshold is more sensitive to changes in fractionalisation than to changes in population inequality.

Our last theorem in this section examines the behaviour of the total rent accumulation in the rent-seeking contest.

Theorem 4.7. *When comparing the total rent accumulated, we obtain the following results:*

(i) Any reorganisation strictly decreases the accumulated rent in which the population distribution is symmetric before the change.

(ii) Any reorganisation strictly decreases the accumulated rent if

$$\frac{(m-k)}{k} \geq k\left(\frac{1}{H_k^{Pre}} - \frac{1+\sum_{i=1}^{k}\Delta_{A_i}^2}{1+k\sum_{i=1}^{k}\Delta_{A_i}^2}\right),$$

and $H_k^{Pre} < \dfrac{k^2}{m^2}$ *when the population distribution is asymmetric before the reorganisation.*

The first part of Theorem 4.7 reaches the same conclusion regarding total accumulated rent as Theorem 4.6 does for rent accumulated in Stage 2. Specifically, it asserts that a symmetric population distribution prior to reorganisation leads to a strict decline in total accumulated rent.

However, the second part of Theorem 4.7 presents a different conclusion. It states that a sufficient condition for the total accumulated rent to strictly decline is twofold: first, the proportional increase in the number of AUs must be greater than or equal to the threshold (which is identical to the one in Theorem 4.6's second part, though it bears noting that this condition is both necessary and sufficient for Stage 2 rent to weakly decline if the post-reorganisation population distribution is symmetric); and second, population inequality must be greater than $\dfrac{(m^2-k^2)}{m^2}$.

Due to the cumulative nature of the Stage 2 rents and the fact that the total rent accumulated is simply the sum of the rents accumulated in both stages of the rent-seeking contest, the intuition behind this result is largely the same as in the case of Theorem 4.6. The reason we require an additional lower bound for population inequality is to compensate the larger effects changes in fractionalisation have on the total rent when compared to Stage 2 rents.

The theoretical insights developed in this section offer some testable hypotheses assuming that the total population before and after reorganisation remains the same[12]. First, a rent-seeking central planner has an unambiguous preference for a proliferatory reorganisation. Second, the welfare loss observed

[12]Recall that the population distribution of (active) AUs in our model refers to the number of economic agents within each AU, which we assume to be sub-administrative units. Therefore, assuming that reorganisation doesn't alter the total number of sub-administrative units across all AUs is reasonable. This is because changes in the total number of sub-administrative units typically occur gradually over time, as establishing new ones entails considerable costs and the creation of new civic centres. For example, after the bifurcation of Uttarakhand from Uttar Pradesh (UP) in 2000, the former got 13 districts, whereas the latter retained 70 districts (GoI, 2000). A new district was added to UP in 2008 (PTI, 2008), following which, four more were added between 2010-11 (Khan, 2010; Srivastava, 2011) whereas the number of districts in Uttarakhand has not changed to date (IGOD, 2025).

due to higher Stage 1 rents can be compensated if the reorganisation creates a sufficiently large number of additional AUs. To the best of our knowledge, there is no data that directly measures the extent of rent-seeking activities undertaken by an administrative unit. However, the wealth accumulated by governing bodies at the administrative and sub-administrative level[13] following a reorganisation can be used as a proxy for rent-seeking.

Empirically, we often observe that most administrative unit reorganisations are proliferatory, and their non-proliferatory nature is primarily attributed to systemic constraints. Furthermore, while political and sociological factors undoubtedly contribute to administrative unit reorganisation, we propose that a general tolerance for rent-seeking - to the extent that it doesn't significantly impact electoral outcomes - also provides a substantial incentive for such divisions.[14] The stark contrast in the number of administrative unit reorganisations between the United States and India, ranked 28th and 96th respectively in the Corruption Perception Index (refer to Transparency International's 2024 CPI for more details), can be considered as a compelling example. Historically, the United States has experienced only three instances of states being formed by directly splitting from an existing state: Kentucky (from Virginia), Maine (from Massachusetts), and West Virginia (again from Virginia). In contrast, India has undergone ten state bifurcations to date.

5. Rent Comparison with Corner Nash Equilibrium

In this section, we present rent comparison results when all AUs do not participate (as part of their equilibrium play) in the rent-seeking contest either before or after the reorganisation. In this context, we focus on the number of active AUs to classify the nature of AU reorganisation in corner Nash equilibrium. We consider the following classification based on this change: a reorganisation is ***effectively expansive*** when $\hat{k} < \hat{m}$, ***effectively neutral*** when $\hat{k} = \hat{m}$, and ***effectively contractive*** when $\hat{k} > \hat{m}$. It is important to note that these classifications based on active AUs can be independent of whether the reorganisation involves an increase, decrease, or no change in the total number of AUs. For instance, a proliferatory reorganisation ($m > k$) might still be effectively neutral ($\hat{k} = \hat{m}$) or even effectively contractive ($\hat{k} > \hat{m}$). Conversely, a non-proliferatory reorganisation ($k = m$) could be effectively expansive ($\hat{k} < \hat{m}$). Moreover, we assume that $N_{\hat{k}}^{Pre} > \hat{k}(\hat{k} - 1)$, and $N_{\hat{m}}^{Post} > \hat{m}(\hat{m} - 1)$.

Let us revisit the notation introduced earlier, which describes population shares in symmetric and

[13] See, for example, Asher and Novosad (2023).

[14] For instance, the existence of a culture of rent-seeking can lead to persistent attitudes tolerating rent-seeking across the society. See Choi and Storr (2019) for more details.

asymmetric distributions before and after reorganisation. When the population distribution of the active AUs is symmetric both before and after reorganisation then $s_{A_i} = \frac{1}{\hat{k}}$ for $i^*_k \leq i \leq k$ and $s_{B_i} = \frac{1}{\hat{m}}$ for $i^*_m \leq i \leq m$. Next, when the population distribution is asymmetric before redistribution, we have $s_{A_i} = \frac{1}{\hat{k}} + \Delta_{A_i}$ for $i^*_k \leq i \leq k$ where $\Delta_{A_{i^*_k}} \geq \Delta_{A_{i^*_k+1}} \geq \cdots \geq \Delta_{A_k}$ such that $\sum_{i=i^*_k}^{k} \Delta_{A_i} = 0$. Similarly, when the population distribution is asymmetric after redistribution, we have $s_{B_i} = \frac{1}{\hat{m}} + \Delta_{B_i}$, where $\Delta_{B_{i^*_m}} \geq \Delta_{B_{i^*_m+1}} \geq \cdots \geq \Delta_{B_{\hat{m}}}$ such that $\sum_{i=i^*_m}^{m} \Delta_{B_i} = 0$.

5.1 Effectively Neutral Reorganisation

In this section, we assume that the reorganisation is effectively neutral, meaning $\hat{m} = \hat{k}$. Since we are interested in scenarios where there are no interior Nash equilibria (in the rent-seeking contest) either before or after the reorganisation, we must have $\hat{k} < k$ and $\hat{m} < m$. Given this, the behaviour of rent accumulation will now also depend on the scale effect of the total population of active AUs along with the fractionalisation and population inequality.

Theorem 5.1. *When comparing the rent accumulated in Stage 1, we obtain the following results:*

(i) Any reorganisation weakly increases the rent accumulated if and only if $N^{Pre}_{\hat{k}} \geq N^{Post}_{\hat{m}}$ when the population distribution is symmetric before and after the reorganisation.

(ii) Any reorganisation strictly decreases the rent accumulated if $N^{Post}_{\hat{m}} \geq N^{Pre}_{\hat{k}}$ when the population distribution before reorganisation is symmetric and asymmetric after it.

(iii) Any reorganisation strictly increases the rent accumulated if $N^{Pre}_{\hat{k}} \geq N^{Post}_{\hat{m}}$ when the population distribution is asymmetric before the reorganisation and symmetric after it.

Intuitively, Theorem 5.1 reveals three key insights regarding the behaviour of rent accumulation in Stage 1 of the rent-seeking contest. The first part of the theorem considers scenarios where the population distribution of active AUs is symmetric both before and after the reorganisation. In this scenario, given that the number of active AUs is the same and the fact that rent accumulation in Stage 1 is inversely proportional to the total active population (as is evident from the Stage 1 rent expression), the average population per active AU serves as the key measure of competitiveness; a lower average population signifies a more competitive contest. Consequently, if the pre-reorganisation average active AU population is higher than the post-reorganisation one, then rent accumulation increases following the reorganisation, and vice versa.

The intuition behind the second and third parts is closely related, with the third part's intuition being symmetrically opposite to that of the second. The second part addresses scenarios where the population distribution of active AUs is symmetric before the reorganisation but becomes asymmetric afterward. In such cases, rent accumulation declines when the average population of active AUs is lower. This outcome occurs because a reduced average population indicates that the population distribution of the active AUs is more concentrated than before. For example, assume there are three active groups. Let the population distribution of the active AUs before the reorganisation be $(6, 6, 6)$, yielding $N_{\hat{k}}^{Pre} = 18$, and after the reorganisation be $(21, 20, 9)$, yielding $N_{\hat{m}}^{Post} = 50$. In this instance, rent-seeking would naturally be more competitive before the reorganisation than after it. Hence, the reorganisation will lead to a strict decline in rent-seeking when $N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre}$.

Note that the above theorem is silent regarding the case where the population distribution of active AUs is asymmetric both before and after the reorganisation. In this case, an effectively neutral reorganisation weakly increases the rent accumulated in Stage 1 of the rent-seeking contest if and only if

$$\left(N_{\hat{k}}^{Pre}\right)^2 \left(1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2\right) \geq \left(N_{\hat{m}}^{Post}\right)^2 \left(1 + \sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right).$$

Theorem 5.2. *When comparing the rent accumulated in Stage 2, we obtain the following results:*

(i) Any reorganisation weakly decreases the rent accumulated if and only if $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$ when the population distribution is symmetric before and after the reorganisation.

(ii) Any reorganisation strictly decreases the rent accumulated if $N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre} \geq H_{\hat{m}}^{Post} N_{\hat{m}}^{Post}$ when the population distribution is symmetric before and asymmetric after the reorganisation.

(iii) Any reorganisation strictly increases the rent accumulated if $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post} \geq H_{\hat{k}}^{Pre} N_{\hat{k}}^{Pre}$ when the population distribution is asymmetric before and symmetric after the reorganisation.

First, note that rent accumulation is directly proportional to the total population size of active AUs as per the Stage 2 rent expressions (11)-(12). Given that the fractionalisation level and population inequality are identical both before and after the reorganisation, the intuition behind the first part of the above theorem is identical to the first part of Theorem 5.1: the average population size of active AUs emerges as the measure of competitiveness, and the reorganisation weakly decreases Stage 2 rents if and only if it lowers the average of the active AU population distribution.

Now, let's discuss the intuition of the second part of the above theorem; the third part's intuition is symmetrically opposite to the second. We already know from Theorem 4.3 that Stage 2 rents decline when the reorganisation converts a symmetric active AU population distribution into an asymmetric one. Since rent-accumulation also depends on the total population of active AUs, to ensure that Stage 2 rents decline, two conditions must be met: first, the average population of active AUs is higher after the reorganisation; and second, the ratio of the average population of active AUs before reorganisation to that after reorganisation must exceed $H_{\hat{m}}^{Post}$.

The preceding theorem does not address scenarios where the population distribution of active AUs remains asymmetric both before and after an effectively neutral reorganisation. In such instances, a sufficient condition for the rent accumulated in Stage 2 of the rent-seeking contest to weakly decrease (weakly increase) is the following:

$$\hat{k}\left(H_{\hat{k}}^{Pre}N_{\hat{k}}^{Pre} - H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}\right) + (\hat{k}-1)\left(\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{m}}^{Post} - \left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)N_{\hat{k}}^{Pre}\right) \geq (\leq) 0.$$

Since $R^{Pre} = R_1^{Pre} + R_2^{Pre}$ and $R^{Post} = R_1^{Post} + R_2^{Post}$, the following theorem is an immediate consequence of Theorems 5.1 and 5.2.

Theorem 5.3. *When comparing the total rent accumulated in the rent-seeking contest, we obtain the following results:*

(i) Any reorganisation weakly decreases (weakly increases) the rent accumulated if $N_{\hat{k}}^{Pre} \geq (\leq) N_{\hat{m}}^{Post}$ when the population distribution is symmetric before and after the reorganisation.

(ii) Any reorganisation strictly decreases the rent accumulated if $N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre} \geq H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}$ when the population distribution is symmetric before and asymmetric after the reorganisation.

(iii) Any reorganisation strictly increases the rent accumulated if $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post} \geq H_{\hat{k}}^{Pre}N_{\hat{k}}^{Pre}$ when the population distribution is asymmetric before and symmetric after the reorganisation.

5.2 Effectively Non-Neutral Reorganisation

In this section, we provide the comparison of rent-accumulated in the rent-seeking contest when the reorganisation is effectively non-neutral, i.e., effectively expansive or effectively contractive. Note that when the reorganisation is effective expansive (effectively contractive) then $\hat{k} < \hat{m} \leq m$ ($\hat{m} < \hat{k} \leq k$, meaning that all AUs can participate in the rent-seeking contest after (before) the reorganisation even though this is not the case before (after) it.

Theorem 5.4. *Any effectively expansive (effectively contractive) reorganisation strictly increases (strictly decreases) the rent accumulated in Stage 1 of the rent-seeking contest if* $N^{Pre}_{\hat{k}} \geq N^{Post}_{\hat{m}}$ ($N^{Post}_{\hat{m}} \geq N^{Pre}_{\hat{k}}$).

The result in the above theorem is connected to Theorem 4.5, stating that the conclusion of the latter theorem holds true for an effectively expansive reorganisation, provided the average active AU population size falls after the reorganisation. For an effectively contractive reorganisation, the above theorem states that the rent accumulation is Stage 1 declines when the average active AU population size rises after the reorganisation.

The underlying intuition for this theorem mirrors that provided for Theorem 4.5 when considering an effectively expansive reorganisation along with the assumption of total population sizes of active AUs, as both indicate an increase in fractionalisation within the population distribution of AUs. However, the above theorem shares a new insight: an effectively contractive reorganisation along with the assumption of total population sizes of active AUs, leads to a decrease in fractionalisation.

Let's elaborate on why this holds true for the specific case where the active AU population distribution is symmetric before an effectively expansive reorganisation and asymmetric after it. Given a symmetric pre-reorganisation population distribution of active AUs, each active AU has a population of $n_{A_i} = \frac{N^{Pre}_{\hat{k}}}{\hat{k}}$ for all $i = i^*_k, \ldots, k$.[15] According to Fact 4, among all post-reorganisation population distributions permitting an interior Nash equilibrium when the total population size is $N^{Post}_{\hat{m}}$ and the number of active AUs is $\hat{m}$, the sum of squares is maximized when $n_{B_i} = \left(\frac{N^{Post}_{\hat{m}} - r_{\hat{m}}}{\hat{m} - 1}\right)$ for $i = i^*_m, \ldots, m-1$, and $n_{B_m} = r_{\hat{m}}$, where $1 \leq r_{\hat{m}} \leq m-1$ is the smallest integer satisfying $(\hat{m}-1)\left(\frac{N - r_{\hat{m}}}{\hat{m} - 1}\right) + r_{\hat{m}} = N^{Post}_{\hat{m}}$ and $r_{\hat{m}} < \left(\frac{N - r_{\hat{m}}}{\hat{m} - 1}\right)$. Since $\hat{m} > \hat{k} > 1$ and $N^{Pre}_{\hat{k}} \geq N^{Post}_{\hat{m}}$, our claim regarding increased fractionalisation follows directly from Lemma B.3[16]. Specifically, comparing the sum of squares:

$$\sum_{i=i^*_m}^{m} n^2_{B_i} = (\hat{m}-1)\left(\frac{N^{Post}_{\hat{m}} - r_{\hat{m}}}{\hat{m}-1}\right)^2 + r^2_{\hat{m}} < \frac{(\hat{m}-1)}{(\hat{k}-1)}\left(\frac{N^{Pre}_{\hat{k}}}{N^{Post}_{\hat{m}}}\right)^2\left(\frac{N^{Pre\,2}_{\hat{k}}}{\hat{k}}\right) = \frac{(\hat{m}-1)}{(\hat{k}-1)}\left(\frac{N^{Pre}_{\hat{k}}}{N^{Post}_{\hat{m}}}\right)^2 \sum_{i=i^*_k}^{k} n^2_{A_i}.$$

This inequality thereby implies an increase in fractionalisation.

[15]Here, we assume that the pair $N^{Pre}_{\hat{k}}$ and $\hat{k}$ admits a symmetric distribution, i.e., $\hat{k}$ perfectly divides $N^{Pre}_{\hat{k}}$.

[16]See Appendix B.

On the other hand, if the reorganisation is effectively contractive then, given $\hat{k} > \hat{m} > 1$ and $N_{\hat{k}}^{Pre} \leq N_{\hat{m}}^{Post}$, our claim regarding decreased fractionalisation follows from Corollary B.1[17]. Comparing the sum of squares in this case:

$$\sum_{i=i_m^*}^{m} n_{B_i}^2 = (\hat{m}-1)\left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right)^2 + r_{\hat{m}}^2 > \frac{(\hat{k}-1)}{(\hat{m}-1)}\left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2\left(\frac{N_{\hat{k}}^{Pre\,2}}{\hat{k}}\right) = \frac{(\hat{k}-1)}{(\hat{m}-1)}\left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2 \sum_{i=1}^{k} n_{A_i}^2.$$

This inequality, in turn, implies a decrease in fractionalisation.

Theorem 5.5. *Let the reorganisation be effectively expansive (effectively contractive) and let $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$ ($N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre}$). When comparing the rent accumulated in Stage 2, we obtain the following results:*

(i) Any reorganisation weakly decreases (weakly increases) the accumulated rent in which the population distribution is symmetric before the change.

(ii) When the population distribution is asymmetric before the reorganisation then:

(a) any reorganisation weakly decreases (weakly increases) the accumulated rent if and only if

$$\frac{\left(\overline{n}_{\hat{k}}^{Pre} - \overline{n}_{\hat{m}}^{Post}\right)}{\overline{n}_{\hat{m}}^{Post}} \geq (\leq) \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2}{1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2}\right),$$

when the population distribution is symmetric after the reorganisation; and

(b) any reorganisation strictly decreases (strictly increases) the accumulated rent if

$$\frac{\left(\overline{n}_{\hat{k}}^{Pre} - \overline{n}_{\hat{m}}^{Post}\right)}{\overline{n}_{\hat{m}}^{Post}} \geq \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2}{1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2}\right),$$

when the population distribution is asymmetric after the reorganisation,

where $\overline{n}_{\hat{k}}^{Pre}$ and $\overline{n}_{\hat{m}}^{Post}$ are the average population size of active AUs before and after the reorganisation respectively.

Theorem 5.5 presents results for Stage 2 rent comparison that are largely similar to those found in Theorem 4.6. However, there are two key distinctions.

[17] See Appendix B.

First, this theorem extends the analysis to include the impact of an effectively contractive reorganisation, a scenario not covered by Theorem 4.6. It demonstrates that the outcomes for an effectively contractive reorganisation are symmetrically opposite to those of an effectively expansive reorganisation (which is analogous to a proliferatory reorganisation).

Second, the threshold condition in part (ii) of this theorem now applies to the proportional decrease in the average population of active AUs, $\frac{\left(\overline{n}_{\hat{k}}^{Pre} - \overline{n}_{\hat{m}}^{Post}\right)}{\overline{n}_{\hat{m}}^{Post}}$, rather than the proportional increase in the number of active AUs. This shift reflects how different population sizes affect rent accumulation in Stage 2.

Before moving to the next theorem, a crucial remark regarding Theorem 5.5 is in order. The condition on the proportional increase in the number of active AUs is sufficient for the results in parts (ii.a) and (ii.b) of the theorem to hold. Let's demonstrate this for an effectively expansive reorganisation, where $\hat{m} > \hat{k} > 1$ and $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$. In this scenario, we have:

$$\frac{\left(\overline{n}_{\hat{k}}^{Pre} - \overline{n}_{\hat{m}}^{Post}\right)}{\overline{n}_{\hat{m}}^{Post}} = \frac{\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\hat{m} - \hat{k}\right)}{\hat{k}} \geq \frac{\left(\hat{m} - \hat{k}\right)}{\hat{k}}.$$

Building on this, the following inequality is sufficient for Stage 2 rents to weakly decline and strictly decline in parts (ii.a) and (ii.b) respectively:

$$\frac{(\hat{m} - \hat{k})}{\hat{k}} \geq \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2}{1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2}\right).$$

A similar argument applies to an effectively contractive reorganisation. In that case, the following inequality is sufficient for Stage 2 rents to weakly decline and strictly decline in parts (ii.a) and (ii.b) respectively:

$$\frac{(\hat{m} - \hat{k})}{\hat{k}} \leq \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2}{1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2}\right).$$

Theorem 5.6. *Let the reorganisation be effectively expansive (effectively contractive) and let* $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$ ($N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre}$)*. When comparing the total rent accumulated, we obtain the following results:*

(i) Any reorganisation strictly decreases (strictly increases) the accumulated rent in which the population distribution is symmetric before the change.

(ii) Any reorganisation strictly decreases (strictly increases) the accumulated rent if

$$\frac{(\hat{m}-\hat{k})}{\hat{k}} \geq (\leq)\hat{k}\left(\frac{1}{H^{Pre}_{\hat{k}}} - \frac{1+\sum_{i=i^*_k}^{k}\Delta^2_{A_i}}{1+\hat{k}\sum_{i=i^*_k}^{k}\Delta^2_{A_i}}\right),$$

and $H^{Pre}_{\hat{k}} < (>)\dfrac{\hat{k}^2}{\hat{m}^2}$ *when the population distribution is asymmetric after the reorganisation.*

The intuition for Theorem 5.6 largely follows from that of Theorem 4.7. Moreover, the sufficient condition outlined in the second part of Theorem 4.7 also proves sufficient for the second part of Theorem 5.6.

6. Concluding Remarks

Building on the seminal work of Katz and Tokatlidu (1996) and Stein and Rapoport (2004), we've examined a two-stage rent-seeking contest. In this setting, administrative units, consisting of economic agents governed by a central planner, compete for a share of a public project's budget. We characterize the conditions under which the central planner opts for reorganisation and analyse the resulting aggregate and stage-wise rent accumulation. Our results underscore that while reorganisation yields private gains for the central planner, social welfare can also be enhanced through careful calibration. This involves either sufficiently increasing the number of post-reorganisation administrative units or, when feasible, transforming a symmetric population distribution into an asymmetric one. Our results, particularly those concerning proliferatory reorganisations, offer several strong testable hypotheses. These insights suggest that a general tolerance for rent-seeking serves as a significant incentive for the observed prevalence of proliferatory administrative unit reorganisations.

Our model provides a robust foundation for several extensions and offers key insights for related literature. One promising avenue is to superimpose a political economy structure onto the current framework, exploring the implications of different administrative regimes, such as a unitary versus a federal structure. Our current model is agnostic to such distinctions, inferring central and local planners' preferences for reorganisation solely based on the total rent they accrue. However, incorporating administrative regimes introduces several complexities:

(i) Rent Distribution among Local Planners: The precise mechanism by which local planners divide the rent extracted in Stage 2 of the rent-seeking contest becomes crucial, especially within a federal administrative regime.

(ii) Planner Incentives Across Reorganisations: The incentives of local planners are also influenced by their roles before and after reorganisation. Pre-reorganisation local planners might represent different administrative units, or they could even cease to be local planners altogether in the new structure.

Addressing these nuances would require an elaborate political economy model. Such a model would allow for a comprehensive examination of how specific administrative regimes impact rent-seeking dynamics in the context of administrative unit reorganisation.

Another promising avenue for future research involves explicitly modelling how the share of funds won by each administrative unit and the economic agent therein (for example, a sub-administrative unit) is utilized. Our current framework assumes the prize (the public project fund) is divisible but remains agnostic about how AUs and their economic agents ultimately use their share, whether for the creation of public, private, or mixed goods. This utilization, however, can critically generate various spillover effects. These spillovers can be both positive (e.g., building a hospital benefiting neighbouring units) and negative (e.g., establishing a polluting waste processing unit). A crucial aspect here is that the ability to internalize these spillover effects might significantly differ based on the ethno-linguistic divisions present among AUs or economic agents; even beneficial spillovers may not be fully realized across such divides. Furthermore, the reorganisation itself might fundamentally alter the preferences of economic agents regarding fund utilization, thereby changing the nature and effect of any resulting spillovers. For example, a unified administrative unit (for example, the erstwhile unified Andhra Pradesh) might prioritize creating large-scale public goods like universities. In contrast, a proliferatory reorganisation could compel smaller, newly formed units (such as a divided Andhra Pradesh and Telangana) to allocate funds towards basic administrative infrastructure like district courts or police stations. Such a shift in priorities would inevitably reshape the type and impact of spillovers.

References

[1] Alesina, A., R. Baqir, and W. Easterly (1999): "Public Goods and Ethnic Divisions," ***The Quarterly Journal of Economics***, 114, 1243–1284.

[2] Asher, S. and P. Novosad (2015): "The Impacts of Local Control over Political Institutions: Evidence from State Splitting in India," ***Unpublished manuscript***.

[3] ——— (2023): "Rent-Seeking and Criminal Politicians: Evidence from Mining Booms," ***The Review of Economics and Statistics***, 105, 20–39.

[4] Atkinson, A. B. (1970): "On the Measurement of Inequality," ***Journal of Economic Theory***, 2, 244–263.

[5] Bhattacharya, P. and J. Rampal (2024): "Contests within and between groups: Theory and experiment," ***Games and Economic Behavior***, 145, 467–492.

[6] Billing, T. (2019): "Government Fragmentation, Administrative Capacity, and Public Goods: The Negative Consequences of Reform in Burkina Faso," ***Political Research Quarterly***, 72, 669–685.

[7] Bolaji, M. and M. A. Gariba (2020): "The Scramble for the Partition of the Northern Region of Ghana," ***African Sociological Review/Revue Africaine de Sociologie***, 24, 75–104.

[8] Choi, J. P., S. M. Chowdhury, and J. Kim (2016): "Group Contests with Internal Conflict and Power Asymmetry," ***The Scandinavian Journal of Economics***, 118, 816–840.

[9] Choi, S. G. and V. H. Storr (2019): "A culture of rent seeking," ***Public Choice***, 181, 101–126.

[10] Cohen, I. (2024): "Documenting Decentralization: Empirical Evidence on Administrative Unit Proliferation from Uganda," ***The World Bank Economic Review***, 38, 772–795.

[11] Collier, P. and A. Hoeffler (1998): "On economic causes of civil war," ***Oxford Economic Papers***, 50, 563–573.

[12] Dasgupta, I. and R. G. Neogi (2018): "Between-group contests over group-specific public goods with within-group fragmentation," ***Public Choice***, 174, 315–334.

[13] Esteban, J., L. Mayoral, and D. Ray (2012): "Ethnicity and Conflict: An Empirical Study," ***American Economic Review***, 102, 1310–1342.

[14] Esteban, J. and D. Ray (2008): "Polarization, Fractionalization and Conflict," ***Journal of Peace Research***, 45, 163–182.

[15] ——— (2011): "Linking Conflict to Inequality and Polarization," ***American Economic Review***, 101, 1345–1374.

[16] Fitrani, F., B. Hofman, and K. Kaiser (2005): "Unity in diversity? The creation of new local governments in a decentralising Indonesia," ***Bulletin of Indonesian Economic Studies***, 41, 57–79.

[17] Garcia Montalvo, J. and M. Reynal-Querol (2002): "Why Ethnic Fractionalization? Polarization, Ethnic Conflict and Growth," Economics Working Papers 660, Department of Economics and Business, Universitat Pompeu Fabra.

[18] GoI (2000): "The Uttar Pradesh Reorganisation Act, 2000," Act No. 29 of 2000, Accessed: 14-07-2025.

[19] Grossman, G. and J. I. Lewis (2014): "Administrative Unit Proliferation," ***American Political Science Review***, 108, 196–217.

[20] Grossman, G., J. H. Pierskalla, and E. Boswell Dean (2017): "Government Fragmentation and Public Goods Provision," ***The Journal of Politics***, 79, 823–840.

[21] Hausken, K. (2005): "Production and Conflict Models Versus Rent-Seeking Models," ***Public Choice***, 123, 59–93.

[22] IGOD (2025): "Districts -Uttarakhand," Accessed : 16-07-2025.

[23] Katz, E. and J. Tokatlidu (1996): "Group competition for rents," ***European Journal of Political Economy***, 12, 599–607.

[24] Khan, A. (2010): "Uttar Pradesh gets one more district," ***The Hindu***, accessed : 14-07-2025.

[25] Kimura, E. (2013): ***Political Change and Territoriality in Indonesia: Provincial Proliferation***, Routledge.

[26] Lewis, B. D. (2017): "Does local government proliferation improve public service delivery? Evidence from Indonesia," ***Journal of Urban Affairs***, 39, 1047–1065.

[27] Lewis, J. I. (2014): "When Decentralization Leads to Recentralization: Subnational State Transformation in Uganda," ***Regional & Federal Studies***, 24, 571–588.

[28] Majeed, A. (2003): "The Changing Politics of States' Reorganization," ***Publius: The Journal of Federalism***, 33, 83–98.

[29] Malesky, E. (2009): "Gerrymandering—Vietnamese Style: Escaping the Partial Reform Equilibrium in a Nondemocratic Regime," ***The Journal of Politics***, 71, 132–159.

[30] Münster, J. and K. Staal (2012): "How organizational structure can reduce rent-seeking," ***Public Choice***, 150, 579–594.

[31] PTI (2008): "CM announces creation of new district in UP," ***India Today***, accessed : 14-07-2025.

[32] Srivastava, P. (2011): "Uttar Pradesh gets three new districts," ***India Today***, accessed : 14-07-2025.

[33] Stein, W. E. and A. Rapoport (2004): "Asymmetric Two-Stage Group Rent-Seeking: Comparison of Two Contest Structures," ***Public Choice***, 118, 151–167.

[34] Tullock, G. (1980): "Efficient rent seeking. jm buchanan, rd tollison, g. tullock, eds., toward a theory of the rent seeking society," ***A & M University Press, College Station, TX: Texas***.

[35] Vaibhav, V. and K. Ramaswamy (2022): "Does the creation of smaller states lead to higher economic growth? Evidence from state reorganization in India," Indira Gandhi Institute of Development Research, Mumbai Working Papers 2022-007, Indira Gandhi Institute of Development Research, Mumbai, India.

[36] Wärneryd, K. (1998): "Distributional conflict and jurisdictional organization," ***Journal of Public Economics***, 69, 435–450.

[37] Yalonetzky, G. (2020): "Inequality of ratios," ***METRON***, 78, 193–217.

Appendix

A. Proofs

Proof of Fact 1. We only prove this fact for the rent-seeking contest before the reorganisation; one can establish the results for the rent-seeking contest after the reorganisation using similar arguments.

First, we solve the second stage of the game. In the second stage of the game, economic agent j in an AU A_i maximizes $U^j_{A_i,2}$ such that $a^j_{i,2} \geq 0$. We focus on symmetric equilibrium, i.e., for all agents j in AU A_i, we assume $a^{j*}_{A_i,2} = a^*_{A_i,2}$. If the AU remains active in the first stage of the contest, then, under a symmetric equilibrium, as mentioned in Stein and Rapoport (33), a corner solution where $a^*_{A_i,2} = 0$ is not observed since the j^{th} economic agent in AU A_i will win the entire share $\theta_{A_1,1}$ by a slight increase in $a^j_{A_i,2}$. At interior equilibrium, we must have $\dfrac{\partial U^j_{A_i,2}}{\partial a^j_{i,2}} = 0$, solving which gives us the Stage 2 equilibrium effort exerted by j^{th} economic agent in the active AU A_i, or

$$a^*_{A_i,2} = \frac{(n_{A_i} - 1)}{n^2_{A_i}}\theta_{A_1,1}$$

If the AU A_i is inactive in the first stage of the contest, then under symmetric equilibrium, $a^*_{A_i,2} = 0$ as there is no share of the pie over which the economic agents can fight in the second stage. Hence, the optimal payoff for j^{th} economic agent in an active AU A_i in Stage 2 pre-reorganisation is given by

$$U^*_{A_i,2} = \frac{\theta_{A_i,1}}{n^2_{A_i}}$$

In the first stage of the game, before reorganisation, j^{th} economic agent in AU A_i maximizes $U^j_{A_i,1} = U^*_{A_i,2} - a^j_{A_i,1}$ such that $a^j_{A_i,1} \geq 0$. In Stage 1 of the rent-contest, in equilibrium we have,

$$\frac{\partial U^j_{A_i,1}}{\partial a^{j*}_{i,1}} \leq 0$$

Here, if $a^{j*}_{i,1} > 0$ then $\dfrac{\partial U^j_{A_i,1}}{\partial a^{j*}_{i,1}} = 0$ (which is the case where the agent j of AU i exerts positive effort in equilibrium) and if $a^{j*}_{i,1} = 0$ then $\dfrac{\partial U^j_{A_i,1}}{\partial a^{j*}_{i,1}} < 0$ (which is the case where the agent j of AU i exerts no effort in equilibrium). In Stage 1, we again focus on symmetric equilibrium within every AU A_i, i.e., we have

$a^{j*}_{A_i,1} = a^*_{A_i,1}$ for all agents j in AU A_i.[18] Fix an equilibrium action profile $(a^*_{1,1}, \ldots, a^*_{k,1})$.[19] Thus, the Stage 1 first order condition, or $\frac{\partial U^j_{A_i,1}}{\partial a^{j*}_{i,1}} \leq 0$ is equivalent to

$$\frac{1}{n^2_{A_i}} \left(\frac{\sum_{i=1}^{k} \sum_{j=1}^{n_{A_i}} a^{j*}_{i,1} - \sum_{j=1}^{n_{A_i}} a^{j*}_{i,1}}{\left(\sum_{i=1}^{k} \sum_{j=1}^{n_{A_i}} a^{j*}_{i,1} \right)^2} \right) \leq 1$$

or

$$n_{A_i} a^*_{i,1} \geq \sum_{i=1}^{k} n_{A_i} a^*_{i,1} - n^2_{A_i} \left(\sum_{i=1}^{k} n_{A_i} a^*_{i,1} \right)^2. \tag{15}$$

Summing this over all i, we get

$$\sum_{i=1}^{k} n_{A_i} a^*_{i,1} \geq \frac{(k-1)}{\sum_{i=1}^{k} n^2_{A_i}}. \tag{16}$$

This means that there exists some AU A_i such that $a^*_{i,1} > 0$ as otherwise from (16), we have

$$0 > \frac{(k-1)}{\sum_{i=1}^{k} n^2_{A_i}} > 0,$$

a contradiction.

Similarly, for AU A_i and A_j with $i < j$, if $a^*_{j,1} = 0$ then it must be the case that $a^*_{i,1} = 0$. Suppose not, i.e., $a^*_{j,1} = 0$ and $a^*_{i,1} > 0$. Since $n_{A_i} \geq n_{A_j}$, using (15), we have

$$0 = n_{A_j} a^*_{j,1} > \sum_{i=1}^{k} n_{A_i} a^*_{i,1} - n^2_{A_j} \left(\sum_{i=1}^{k} n_{A_i} a^*_{i,1} \right)^2 \geq \sum_{i=1}^{k} n_{A_i} a^*_{i,1} - n^2_{A_i} \left(\sum_{i=1}^{k} n_{A_i} a^*_{i,1} \right)^2 = n_{A_i} a^*_{i,1} > 0,$$

a contradiction.

Therefore, we have an AU with index i^*_k such that for all AU's with index $j < i^*_k$, $a^*_{j,1} = 0$ and for all AU's with index $j \geq i^*_k$, $a^*_{j,1} > 0$. Also, it is easy to establish that $i^*_k \leq k - 1$; otherwise, if $i^*_k = k$ then the AU with index $k - 1$ can slightly increase their effort level and gain positive utility. ■

Proof of Fact 2. (i) In this section, we derive the rent expression in Stage 1 of the rent-seeking contest

[18] See (33) for details.

[19] For notational convenience, we abuse the notation here; for every AU i, we write the effort level of a representative symmetric action of the agents as shorthand for the (symmetric) action profile of the agents in that AU.

before reorganisation. From Fact 1, there exists i_k^* such that for all AUs with index $i \geq i_k^*$, $a_{i,1}^* > 0$, and for all AUs with index $i < i_k^*$, $a_{i,1}^* = 0$. Hence, rent accumulated in Stage 1 before reorganisation can be re-written as $R_1^{Pre} = \sum_{i=i_k^*}^{k} n_{A_i} a_{i,1}^*$. Now, using the equality in (15), summing it from i_k^* to k and rewriting the resultant expression in terms of shares of the AU population, we get

$$R_1^{Pre} = \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} n_{A_i}^2} = \frac{(k - i_k^*)}{\left(N_{\hat{k}}^{Pre}\right)^2 \sum_{i=i_k^*}^{k} s_{A_i}^2}.$$

Using similar arguments, one can derive the rent expression in Stage 1 of the rent-seeking contest after reorganisation.

(ii) Now, we move on to the derivation of the Stage 2 rent expression before reorganisation. Again using Fact 1, rent generated in Stage 2 is given by $R_2^{Pre} = \sum_{i=i_k^*}^{k} n_{A_i} a_{i,2}^* = \sum_{i=i_k^*}^{k} \frac{(n_{A_i} - 1)}{n_{A_i}} \theta_{A_i,1}$, where, under a symmetric equilibrium, $\theta_{A_i,1} = \frac{n_{A_i} a_{i,1}^*}{\sum_{i=i_k^*}^{k} n_{A_i} a_{i,1}^*}$ (Equation 6). Also, observe that $\sum_{i=1}^{k} n_{A_i} a_{i,1}^* = \sum_{i=i_k^*}^{k} n_{A_i} a_{i,1}^* = R_1^{Pre}$ since $a_{i,1}^* = 0\ \forall i < i_k^*$ (Fact 1). Thus, using the equality in 15 and the expression for R_1^{Pre} obtained in part (i), for all $i \geq i_k^*$, the Stage 1 equilibrium effort exerted by j^{th} economic agent in AU A_i, or $a_{i,1}^*$ is given by

$$a_{i,1}^* = \frac{(k - i_k^*)}{n_{A_i} \left(\sum_{i=i_k^*}^{k} n_{A_i}^2 \right)^2} \left(\sum_{i=i_k^*}^{k} n_{A_i}^2 - (k - i_k^*) n_{A_i}^2 \right)$$

Substituting the expressions for $a_{i,1}^*$ and R_1^{Pre} (derived in part (i)) in $\theta_{A_i,1}$, R_2^{Pre} can be restated as follows

$$R_2^{Pre} = \sum_{i=i_k^*}^{k} n_{A_i} a_{i,2}^* = \sum_{i=i_k^*}^{k} \frac{(n_{A_i} - 1)}{n_{A_i}} \theta_{A_i,1} = \frac{1}{\sum_{i=i_k^*}^{k} n_{A_i}^2} \sum_{i=i_k^*}^{k} \left[\left(1 - \frac{1}{n_{A_i}} \right) \left(\sum_{i=i_k^*}^{k} n_{A_i}^2 - (k - i_k^*) n_{A_i}^2 \right) \right]$$

or,

$$R_2^{Pre} = \frac{1}{\sum_{i=i_k^*}^{k} n_{A_i}^2}\left[\sum_{i=i_k^*}^{k} n_{A_i}^2 + (k - i_k^*)\sum_{i=i_k^*}^{k} n_{A_i} - \sum_{i=i_k^*}^{k}\frac{1}{n_{A_i}}\Big(\sum_{i=i_k^*}^{k} n_{A_i}^2\Big)\right] = \left[1 + \frac{(k - i_k^*)\sum_{i=i_k^*}^{k} n_{A_i}}{\sum_{i=i_k^*}^{k} n_{A_i}^2} - \sum_{i=i_k^*}^{k}\frac{1}{n_{A_i}}\right]$$

Writing R_2^{Pre} in terms of population shares, we have

$$R_2^{Pre} = \left[1 + \frac{(k - i_k^*)N_{\hat{k}}^{Pre}\sum_{i=i_k^*}^{k} s_{A_i}}{\left(N_{\hat{k}}^{Pre}\right)^2\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k}\frac{1}{N_{\hat{k}}^{Pre} s_{A_i}}\right] = \frac{1}{N_{\hat{k}}^{Pre}}\left[N_{\hat{k}}^{Pre} + \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k}\frac{1}{s_{A_i}}\right]$$

Using similar arguments, one can derive the rent expression in Stage 2 of the rent-seeking contest after reorganisation. ■

Proof of Fact 3. For AU A_i, with $i \geq i_k^*$, to participate in Stage 1 of the rent-seeking contest, given $a_{h,1}^* > 0\ \forall h \geq i_k^*,\ h \neq i$, and $a_{h,1}^* = 0\ \forall h < i_k^*$, we must have $\left.\frac{\partial U_{A_{i,1}}^j}{\partial a_{i,1}^*}\right|_{a_{i,1}^*=0} > 0$, or

$$\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^* > n_{A_i}^2\Big(\sum_{h=i_k^*}^{k} n_{A_h} a_{h,1}^*\Big)^2.$$

or

$$\frac{1}{\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^*} > n_{A_i}^2$$

Observe that the last inequality holds because $a_{i,1}^* = 0$, which implies that $\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^* = \sum_{h=i_k^*}^{k} n_{A_h} a_{h,1}^*$.

Moreover, from Fact 2(i), we have

$$\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^* = \frac{(k - i_k^* - 1)}{\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h}^2}$$

Substituting the expression for $\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^*$ obtained in the previous step in $\frac{1}{\sum_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h} a_{h,1}^*} > n_{A_i}^2$, we have the following

$$\frac{\sum\limits_{\substack{h=i_k^*\\h\neq i}}^{k} n_{A_h}^2}{(k-i_k^*-1)} > n_{A_i}^2,$$

or

$$\sum_{h=i_k^*}^{k} n_{A_h}^2 - (k-i_k^*)n_{A_i}^2 > 0,$$

as required. Hence, for all AUs A_i with $i_k^* \leq i < k$ to participate in the contest such that the total number of active AUs is given by $\hat{k} = k - i_k^* + 1$, $\sum\limits_{h=i_k^*}^{k} n_{A_h}^2 - (k-i_k^*)n_{A_i}^2 = \sum\limits_{h=i_k^*}^{k} n_{A_h}^2 - (\hat{k}-1)n_{A_i}^2 > 0$ must hold for all $i = i_k^*, \ldots, k$. Similarly, for all AUs B_i with $i_m^* \leq i < m$ to participate in the contest such that the total number of active AUs is given by $\hat{m} = m - i_m^* + 1$, $\sum\limits_{h=i_m^*}^{m} n_{B_h}^2 - (m-i_m^*)n_{B_i}^2 = \sum\limits_{h=i_m^*}^{m} n_{B_h}^2 - (\hat{m}-1)n_{B_i}^2 > 0$ must hold for all $i = i_m^*, \ldots, m$. ■

Proof of Theorem 3.1. Using (9), (10), and the facts that $k - i_k^* = \hat{k} - 1$ and $m - i_m^* = \hat{m} - 1$, we know

$$R_1^{Pre} = \frac{(k-i_k^*)}{\left(N_{\hat{k}}^{Pre}\right)^2 \sum_{i=i_k^*}^{k} s_{A_i}^2} = \frac{(\hat{k}-1)}{\left(N_{\hat{k}}^{Pre}\right)^2 \sum_{i=i_k^*}^{k} s_{A_i}^2} = \frac{\hat{k}}{\left(N_{\hat{k}}^{Pre}\right)^2}\omega_{\hat{k}}^{Pre},$$

and

$$R_1^{Post} = \frac{(m-i_m^*)}{\left(N_{\hat{m}}^{Post}\right)^2 \sum_{i=i_m^*}^{k} s_{B_i}^2} = \frac{(\hat{m}-1)}{\left(N_{\hat{m}}^{Post}\right)^2 \sum_{i=i_m^*}^{m} s_{B_i}^2} = \frac{\hat{m}}{\left(N_{\hat{m}}^{Post}\right)^2}\omega_{\hat{m}}^{Post}.$$

Therefore,

$$R_1^{Post} \geq (\leq) R_1^{Pre} \Leftrightarrow \frac{\hat{m}\omega_{\hat{m}}^{Post}}{\hat{k}\omega_{\hat{k}}^{Pre}} \geq (\leq) \left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2,$$

as required. ■

Proof of Theorem 3.2. From (11) and (12), we have

$$R_2^{Pre} = \frac{1}{N_{\hat{k}}^{Pre}}\left[N_{\hat{k}}^{Pre} + \frac{(k-i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}\right],$$

and

$$R_2^{Post} = \frac{1}{N_{\hat{m}}^{Post}}\left[N_{\hat{m}}^{Post} + \frac{(m-i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}\right].$$

Therefore, $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$\frac{1}{N_{\hat{k}}^{Pre}}\left[N_{\hat{k}}^{Pre} + \frac{(k - i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}\right] \geq (\leq) \frac{1}{N_{\hat{m}}^{Post}}\left[N_{\hat{m}}^{Post} + \frac{(m - i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}\right].$$

Since $k - i_k^* = \hat{k} - 1$ and $m - i_m^* = \hat{m} - 1$, $R_2^{Pre} \geq (\leq) R_2^{Post}$ is equivalent to

$$\frac{1}{N_{\hat{k}}^{Pre}}\left[\frac{(\hat{k} - 1)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}\right] \geq (\leq) \frac{1}{N_{\hat{m}}^{Post}}\left[\frac{(\hat{m} - 1)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}}\right],$$

or

$$\frac{1}{N_{\hat{m}}^{Post}}\left[\sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} - \frac{(\hat{m} - 1)}{N_{\hat{m}}^{Post} \sum_{i=i_m^*}^{m} s_{B_i}^2}\right] \geq (\leq) \frac{1}{N_{\hat{k}}^{Pre}}\left[\sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} - \frac{(\hat{k} - 1)}{N_{\hat{k}}^{Pre} \sum_{i=i_k^*}^{k} s_{A_i}^2}\right].$$

Recall that, for all $i = i_k^*, \ldots, k$, $s_{A_i} = \dfrac{1}{\hat{k}} + \Delta_{A_i}$. Then, the arithmetic mean of the population shares of AUs before the reorganisation is given by

$$\frac{1}{\hat{k}} \sum_{i=i_k^*}^{k} s_{A_i} = \frac{1}{\hat{k}},$$

and their harmonic mean is given by

$$\frac{\hat{k}}{\sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}}}.$$

Since $H_{\hat{k}}^{Pre}$ is the ratio of the harmonic to the arithmetic mean of the population shares of AUs before the reorganisation, we have

$$H_{\hat{k}}^{Pre} = \frac{\hat{k}^2}{\sum_{i=i_k^*}^{\hat{k}} \frac{1}{s_{A_i}}},$$

and cross-multiplying, we have

$$\sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} = \frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}.$$

Similarly, we have

$$\sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} = \frac{\hat{m}^2}{H_{\hat{m}}^{Post}}.$$

Substituting these expressions, we have $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$\frac{1}{N_{\hat{m}}^{Post}} \left[\frac{\hat{m}^2}{H_{\hat{m}}^{Post}} - \frac{\hat{m}(\hat{m}-1)}{\hat{m} \sum_{i=i_m^*}^{m} s_{B_i}^2} \right] \geq (\leq) \frac{1}{N_{\hat{k}}^{Pre}} \left[\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}} - \frac{\hat{k}(\hat{k}-1)}{\hat{k} \sum_{i=i_k^*}^{k} s_{A_i}^2} \right],$$

or

$$\frac{\hat{m}}{N_{\hat{m}}^{Post}} \left[\frac{m}{H_m^{Post}} - \frac{(\hat{m}-1)}{\hat{m} \sum_{i=i_m^*}^{m} s_{B_i}^2} \right] \geq (\leq) \frac{\hat{k}}{N_{\hat{k}}^{Pre}} \left[\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \frac{(\hat{k}-1)}{\hat{k} \sum_{i=i_k^*}^{k} s_{A_i}^2} \right],$$

or

$$\frac{\hat{m}}{N_{\hat{m}}^{Post}} \left(\tau_{\hat{m}}^{Post} - \omega_{\hat{m}}^{Post} \right) \geq (\leq) \frac{\hat{k}}{N_{\hat{k}}^{Pre}} \left(\tau_{\hat{k}}^{Pre} - \omega_{\hat{k}}^{Pre} \right),$$

or

$$\frac{\hat{m} \left(\tau_{\hat{m}}^{Post} - \omega_{\hat{m}}^{Post} \right)}{\hat{k} \left(\tau_{\hat{k}}^{Pre} - \omega_{\hat{k}}^{Pre} \right)} \geq (\leq) \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}},$$

as required. ■

Proof of Theorem 3.3. Note that $R^{Pre} \geq (\leq) R^{Post}$ if and only if

$$\frac{1}{N_{\hat{k}}^{Pre}} \left[N_{\hat{k}}^{Pre} + \frac{(k-i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}} \right) - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} \right] \geq (\leq) \frac{1}{N_{\hat{m}}^{Post}} \left[N_{\hat{m}}^{Post} + \frac{(m-i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}} \right) - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} \right],$$

or

$$\frac{1}{N_{\hat{k}}^{Pre}} \left[\frac{(k-i_k^*)}{\sum_{i=i_k^*}^{k} s_{A_i}^2} \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}} \right) - \sum_{i=i_k^*}^{k} \frac{1}{s_{A_i}} \right] \geq (\leq) \frac{1}{N_{\hat{m}}^{Post}} \left[\frac{(m-i_m^*)}{\sum_{i=i_m^*}^{m} s_{B_i}^2} \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}} \right) - \sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} \right].$$

Since $k - i_k^* = \hat{k} - 1$ and $m - i_m^* = \hat{m} - 1$, $R^{Pre} \geq (\leq) R^{Post}$ is equivalent to

$$\frac{1}{N_{\hat{m}}^{Post}} \left[\sum_{i=i_m^*}^{m} \frac{1}{s_{B_i}} - \frac{\hat{m}(\hat{m}-1)}{\hat{m} \sum_{i=i_m^*}^{m} s_{B_i}^2} \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}} \right) \right] \geq (\leq) \frac{1}{N_{\hat{k}}^{Pre}} \left[\sum_{i=1}^{k} \frac{1}{s_{A_i}} - \frac{\hat{k}(\hat{k}-1)}{\hat{k} \sum_{i=1}^{k} s_{A_i}^2} \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}} \right) \right],$$

or

$$\frac{1}{N_{\hat{m}}^{Post}} \left[\frac{\hat{m}^2}{H_{\hat{m}}^{Post}} - \frac{\hat{m}(\hat{m}-1)}{\hat{m} \sum_{i=i[_m*]}^{m} s_{B_i}^2} \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}} \right) \right] \geq (\leq) \frac{1}{N_{\hat{k}}^{Pre}} \left[\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}} - \frac{\hat{k}(\hat{k}-1)}{\hat{k} \sum_{i=i_k^*}^{k} s_{A_i}^2} \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}} \right) \right],$$

or

$$\frac{\hat{m}}{N_{\hat{m}}^{Post}} \left[\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{\hat{m} \sum_{i=1}^{m} s_{B_i}^2} \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}} \right) \right] \geq (\leq) \frac{\hat{k}}{N_{\hat{k}}^{Pre}} \left[\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \frac{(\hat{k}-1)}{\hat{k} \sum_{i=1}^{k} s_{A_i}^2} \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}} \right) \right].$$

Substituting the expressions of $\omega_{\hat{k}}^{Pre}$, $\omega_{\hat{m}}^{Post}$, $\tau_{\hat{k}}^{Pre}$, and $\tau_{\hat{m}}^{Post}$ in the above, we have $R^{Pre} \geq (\leq) R^{Post}$ if and only if

$$\frac{\hat{m}}{N_{\hat{m}}^{Post}}\left(\tau_{\hat{m}}^{Post} - \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\omega_{\hat{m}}^{Post}\right) \geq (\leq) \frac{\hat{k}}{N_{\hat{k}}^{Pre}}\left(\tau_{\hat{k}}^{Pre} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\omega_{\hat{k}}^{Pre}\right),$$

or

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \left(\frac{N_{\hat{m}}+1}{N_{\hat{m}}^{Post}}\right)\omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\omega_{\hat{k}}^{Pre}\right)} \geq (\leq) \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}},$$

as required. ■

We present a unified proof strategy for the results in both Section 4 and Section 5. To this end, we will now consider an effectively neutral, expansive and contractive reorganisation to be one where $\hat{k} \leq k$ and $\hat{m} \leq m$ when proving Theorems 5.1-5.6. The specific results for Section 4 can then be directly obtained by considering the following special cases:

(i) a non-proliferatory reorganisation can be viewed as an effectively neutral reorganisation when setting $\hat{k} = k = \hat{m}$ and $N_{\hat{k}}^{Pre} = N_{\hat{m}}^{Post} = N$, and

(ii) a proliferatory reorganisation is an instance of an effectively expansive reorganisation when setting $\hat{k} = k < m = \hat{m}$ and $N_{\hat{k}}^{Pre} = N_{\hat{m}}^{Post} = N$.

Proof of Theorem 5.1. From Theorem 3.1, we know that $R_1^{Post} \geq (\leq) R_1^{Pre}$ if and only if

$$\frac{\hat{m}\omega_{\hat{m}}^{Post}}{\hat{k}\omega_{\hat{k}}^{Pre}} \geq (\leq) \left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2$$

where $\omega_{\hat{k}}^{Pre} = \dfrac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}$, and $\omega_{\hat{m}}^{Post} = \dfrac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}$.

Rearranging and applying the fact that $\hat{m} = \hat{k}$, we have $R_1^{Post} \geq R_1^{Pre}$ if and only if

$$1 \geq (\leq) \left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2 \frac{\left(1 + \hat{m}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right)}{\left(1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2\right)},$$

or,

$$\left(N_{\hat{k}}^{Pre}\right)^2\left(1 + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2\right) \geq (\leq) \left(N_{\hat{m}}^{Post}\right)^2\left(1 + \sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right), \tag{17}$$

as required.

Next, we show (i). If the population distribution is symmetric before and after the reorganisation then $\sum_{i=i_k^*}^{k} \Delta_{A_i}^2 = 0 = \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$. Therefore, using (17), we have $R_1^{Post} \geq (\leq) R_1^{Pre}$ if and only if $\left(N_{\hat{k}}^{Pre}\right)^2 \geq (\leq) \left(N_{\hat{m}}^{Post}\right)^2$ and since $N_{\hat{k}}^{Pre}, N_{\hat{m}}^{Post} > 0$, $\left(N_{\hat{k}}^{Pre}\right)^2 \geq (\leq) \left(N_{\hat{m}}^{Post}\right)^2$ if and only if $N_{\hat{k}}^{Pre} \geq (\leq) N_{\hat{m}}^{Post}$, thereby implying that $R_1^{Post} \geq R_1^{Pre}$ if and only if $N_{\hat{k}}^{Pre} \geq (\leq) N_{\hat{m}}^{Post}$.

Lastly, we are only left to establish (ii); (iii) can established by arguments symmetric to the ones used to establish (ii). If the population distribution is symmetric before the reorganisation and asymmetric after it then $\sum_{i=i_k^*}^{k} \Delta_{A_i}^2 = 0 < \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$. If $N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre} (> 0)$, we have

$$\left(N_{\hat{m}}^{Post}\right)^2 \left(1 + \sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right) > \left(N_{\hat{m}}^{Post}\right)^2 \geq \left(N_{\hat{k}}^{Pre}\right)^2 = \left(N_{\hat{k}}^{Pre}\right)^2 \left(1 + \sum_{i=i_k^*}^{k} \Delta_{B_i}^2\right),$$

thereby implying that $R_1^{Pre} > R_1^{Post}$ as required. ■

Proof of Theorem 5.2. From Theorem 3.2, we know that $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \omega_{\hat{k}}^{Pre}\right)} \geq (\leq) \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}},$$

where $\omega_{\hat{k}}^{Pre} = \dfrac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}$, $\omega_{\hat{m}}^{Post} = \dfrac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}$, $\tau_{\hat{k}}^{Pre} = \dfrac{\hat{k}}{H_{\hat{k}}^{Pre}}$ and $\tau_{\hat{m}}^{Post} = \dfrac{\hat{m}}{H_{\hat{m}}^{Post}}$.

Rearranging and applying the fact that $\hat{m} = \hat{k}$, we have $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$N_{\hat{k}}^{Pre}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{\left(1 + \hat{m}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right)}\right) \geq (\leq) N_{\hat{m}}^{Post}\left(\frac{\hat{k}}{H_{\hat{k}}^{Post}} - \frac{\left(\hat{k}-1\right)}{\left(1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2\right)}\right),$$

or

$$\hat{k}\left(\frac{1}{H_{\hat{m}}^{Post}} - \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\frac{1}{H_{\hat{k}}^{Pre}}\right) \geq (\leq)(\hat{k}-1)\left(\frac{1}{\left(1 + \hat{k}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2\right)} - \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\frac{1}{\left(1 + \hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2\right)}\right),$$

or,

$$\hat{k}\left(\frac{H_{\hat{k}}^{Pre}N_{\hat{k}}^{Pre}-H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}}{H_{\hat{k}}^{Pre}H_{\hat{m}}^{Post}N_{\hat{k}}^{Pre}}\right)+(\hat{k}-1)\left(\frac{\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{m}}^{Post}-\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)N_{\hat{k}}^{Pre}}{\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{k}}^{Pre}}\right)\geq(\leq)0.$$

Since $N_{\hat{k}}^{Pre}>0$, $0<H_{\hat{k}}^{Pre},H_{\hat{m}}^{Post}\leq 1$, $\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right),\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)\geq 1$, $H_{\hat{k}}^{Pre}H_{\hat{m}}^{Post}N_{\hat{k}}^{Pre}\leq N_{\hat{k}}^{Pre}\leq \left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{k}}^{Pre}$, and therefore, if

$$\hat{k}\left(H_{\hat{k}}^{Pre}N_{\hat{k}}^{Pre}-H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}\right)+(\hat{k}-1)\left(\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{m}}^{Post}-\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)N_{\hat{k}}^{Pre}\right)\geq(\leq)0, \tag{18}$$

then we have $R_2^{Pre}\geq(\leq)R_2^{Post}$.

We are now in a position to prove (i). Since the population distribution is symmetric before and after reorganisation, we have $H_{\hat{k}}^{Pre}=H_{\hat{m}}^{Post}=1$ and $\sum_{i=i_k^*}^{k}\Delta_{A_i}^2=\sum_{i=i_m^*}^{m}\Delta_{B_i}^2=0$, and therefore, using (18),

$$\hat{k}\left(N_{\hat{k}}^{Pre}-N_{\hat{m}}^{Post}\right)+(\hat{k}-1)\left(N_{\hat{m}}^{Post}-N_{\hat{k}}^{Pre}\right)=N_{\hat{k}}^{Pre}-N_{\hat{m}}^{Post}\geq(\leq)0$$

thereby implying that $R_2^{Pre}\geq(\leq)R_2^{Post}$ if and only if $N_{\hat{k}}^{Pre}\geq(\leq)N_{\hat{m}}^{Post}$.

Lastly, we are only left to prove (ii); (iii) can be established using arguments symmetric to the ones used to establish (ii). Since the population distribution is symmetric before the reorganisation and asymmetric after the reorganisation, we have $H_{\hat{k}}^{Pre}=1>H_{\hat{m}}^{Post}$ and $\sum_{i=i_k^*}^{k}\Delta_{A_i}^2=0<\sum_{i=i_m^*}^{m}\Delta_{B_i}^2$. If $N_{\hat{m}}^{Post}\geq N_{\hat{k}}^{Pre}\geq H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}$, we have $\left(1+\hat{k}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2\right)N_{\hat{m}}^{Post}>N_{\hat{m}}^{Post}\geq N_{\hat{k}}^{Pre}=H_{\hat{k}}^{Pre}N_{\hat{k}}^{Pre}\geq H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}$ and therefore, using (18), $R_2^{Pre}>R_2^{Post}$ as required. ■

Proof of Theorem 5.4. We only prove the theorem when the reorganisation is effectively expansive; we can establish the part of the theorem regarding effectively contractive reorganisations using symmetric arguments.

Using Theorem 3.1, we know that $R_1^{Post} > R_1^{Pre}$ is equivalent to

$$\frac{\hat{m}\omega_{\hat{m}}^{Post}}{\hat{k}\omega_{\hat{k}}^{Pre}} > \left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2.$$

Substituting the expressions of ω_m^{Post} and ω_k^{Pre}, $R_1^{Post} > R_1^{Pre}$ is equivalent to

$$\frac{\sum_{i=i_m^*}^{m} s_{B_i}^2}{\sum_{i=i_k^*}^{k} s_{A_i}^2} < \frac{(\hat{m}-1)}{(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2,$$

or,

$$\frac{\sum\limits_{i=1}^{m} s_{B_i}^2}{\sum\limits_{i=i_k^*}^{k} s_{A_i}^2} < \frac{(\hat{m}-1)}{(\hat{k}-1)}.$$

Since $\sum\limits_{i=i_k^*}^{k} s_{A_i}^2 = \frac{1}{\hat{k}}\left(1+\hat{k}\sum\limits_{i=i_k^*}^{k}\Delta_{A_i}^2\right)$ and $\sum\limits_{i=i_m^*}^{m} s_{B_i}^2 = \frac{1}{\hat{m}}\left(1+\hat{m}\sum\limits_{i=i_m^*}^{m}\Delta_{B_i}^2\right)$, we have $R_1^{Post} > R_1^{Pre}$ is equivalent to

$$\frac{1+\hat{m}\sum\limits_{i=i_m^*}^{m}\Delta_{B_i}^2}{1+\hat{k}\sum\limits_{i=i_k^*}^{k}\Delta_{A_i}^2} < \frac{\hat{m}(\hat{m}-1)}{\hat{k}(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2.$$

Since the reorganisation is effectively expansive, we have $\hat{m} > \hat{k} > 1$. Observe that since for all $i = i_k^*, \ldots, k$, $\Delta_{A_i}^2 \geq 0$, we have $1+\hat{k}\sum\limits_{i=i_k^*}^{k}\Delta_{A_i}^2 \geq 1$. Similarly, it must be the case that $1+\hat{m}\sum\limits_{i=i_m^*}^{m}{\Delta_{B_i}}^2 \geq 1$. Further, observe that since $\hat{m} > \hat{k} > 1$ and $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$, $\hat{m}(\hat{m}-1) > \hat{k}(\hat{k}-1)$ implying that $\frac{\hat{m}(\hat{m}-1)}{\hat{k}(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2 > 1$.

If the population distribution after the reorganisation is symmetric, i.e., $\Delta_{B_i} = 0$ for all $i = 1, \ldots, m$, we have

$$\frac{1+\hat{m}\sum\limits_{i=i_m^*}^{m}\Delta_{B_i}^2}{1+\hat{k}\sum\limits_{i=i_k^*}^{k}\Delta_{A_i}^2} = \frac{1}{1+\hat{k}\sum\limits_{i=i_k^*}^{k}\Delta_{A_i}^2} \leq 1 < \frac{\hat{m}(\hat{m}-1)}{\hat{k}(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2,$$

and therefore, we have $R_1^{Post} > R_1^{Pre}$ as required.

Alternatively, assume that the population distribution after reorganisation is asymmetric. We consider two cases:

Case 1: If population distribution is symmetric before reorganisation, then

$$\frac{1 + \hat{m} \sum_{i=i_m^*}^{m} \Delta_{B_i}^2}{1 + \hat{k} \sum_{i=i_k^*}^{k} \Delta_{A_i}^2} = \frac{1 + \hat{m} \sum_{i=i_m^*}^{m} \Delta_{B_i}^2}{1} < \frac{\hat{m}(\hat{m}-1)}{\hat{k}(\hat{k}-1)} \leq \frac{\hat{m}(\hat{m}-1)}{\hat{k}(\hat{k}-1)} \left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}} \right)^2,$$

and therefore, we have $R_1^{Post} > R_1^{Pre}$ as required. Note that the last inequality above is due to Lemma B.3 (when expressing the inequality there in terms of shares of the population).

Case 2: If population distribution is asymmetric before reorganisation then we (hypothetically) break down the reorganisation into two steps as shown below:

$$\text{Asymmetric } \hat{k} \longrightarrow \text{Symmetric } \hat{k} \longrightarrow \text{Asymmetric } \hat{m}$$

In Step 1, we study rent accumulation in Stage 1 when the number of active AUs is constant at $\hat{k}$ both before and after reorganisation but changes the nature of the population distribution of active AUs from asymmetric to symmetric (thereby implying that $\sum_{i=i_k^*}^{k} \Delta_{A_i}^2 > 0 = \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$) and therefore, using Theorem 4.2, we know that the total rent generated in Stage 1 of the game increases after the reorganisation. In Step 2, we study rent accumulation in Stage 1 when the number of AUs (strictly) increases from $\hat{k}$ to $\hat{m}$ and the nature of the population distribution changes from symmetric to asymmetric and we have already established in Case 1 that the rent accumulated in Stage 1 of the game increases in Step 2 as well. Combining these findings, we establish that the total rent generated in Stage 1 is higher after reorganisation when the population distribution is asymmetric before reorganisation (and after reorganisation). ■

Proof of Theorem 5.5. From Theorem 3.2, we know that $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \omega_{\hat{k}}^{Pre}\right)} \geq (\leq) \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}},$$

where $\omega_{\hat{k}}^{Pre} = \dfrac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}$, $\omega_{\hat{m}}^{Post} = \dfrac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}$, $\tau_{\hat{k}}^{Pre} = \dfrac{\hat{k}}{H_{\hat{k}}^{Pre}}$, and $\tau_{\hat{m}}^{Post} = \dfrac{\hat{m}}{H_{\hat{m}}^{Post}}$. Substituting these expressions and cross multiplying, we have $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}\right) \geq (\leq) N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \frac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}\right).$$

Since $\sum_{i=i_k^*}^{k} s_{A_i}^2 = \dfrac{1}{\hat{k}} + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2$ and $\sum_{i=i_m^*}^{m} s_{B_i}^2 = \dfrac{1}{\hat{m}} + \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$, we have $R_2^{Pre} \geq (\leq) R_2^{Post}$ if and only if

$$N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2}\right) \geq (\leq) N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \frac{(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k} \Delta_{A_i}^2}\right). \tag{19}$$

To prove (i), since the population distribution is symmetric before the reorganisation, observe that $H_{\hat{k}}^{Pre} = 1 \geq H_{\hat{m}}^{Post}$ and $\sum_{i=i_k^*}^{k} \Delta_{A_i}^2 = 0 \leq \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$. Also, note that

$$\frac{\hat{m}}{H_{\hat{m}}^{Post}} \geq \hat{m} > \hat{m}-1 \geq \frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2},$$

thereby implying that,

$$\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m} \Delta_{B_i}^2}\right) > 1.$$

Therefore, using (19) and the assumption that $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$, we have

$$N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2}\right) > N_{\hat{k}}^{Pre}\hat{m} > N_{\hat{k}}^{Pre}\hat{k} \geq N_{\hat{m}}^{Post}\hat{k} = N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \frac{(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right),$$

thereby implying that $R_2^{Pre} > R_2^{Post}$ as required.

Next, we prove (ii.a). Since the population distribution is asymmetric before the reorganisation and symmetric after it, observe that $H_{\hat{k}}^{Pre} < 1 = H_{\hat{m}}^{Post}$ and $\sum_{i=i_k^*}^{k}\Delta_{A_i}^2 > 0 \sum_{i=i_m^*}^{m}\Delta_{B_i}^2$. Substituting these expressions in (19), we have $R_2^{Post} \geq (\leq) R_2^{Pre}$ if and only if

$$N_{\hat{k}}^{Pre}\left(\hat{m}^2 - \hat{m}(\hat{m}-1)\right) \geq (\leq) N_{\hat{m}}^{Post}\left(\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}} - \frac{\hat{k}(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right),$$

or

$$N_{\hat{k}}^{Pre}\hat{m} - N_{\hat{m}}^{Post}\hat{k} \geq (\leq) N_{\hat{m}}^{Post}\left(\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}} - \frac{\hat{k}(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2} - \hat{k}\right),$$

or,

$$\frac{\left(N_{\hat{k}}^{Pre}\hat{m} - N_{\hat{m}}^{Post}\hat{k}\right)}{N_{\hat{m}}^{Post}\hat{k}} \geq (\leq) \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right).$$

Using the fact that $\overline{n}_{\hat{k}}^{Pre} = \frac{N_{\hat{k}}^{Pre}}{\hat{k}}$ and $\overline{n}_{\hat{m}}^{Post} = \frac{N_{\hat{m}}^{Post}}{\hat{m}}$, we have $R_2^{Post} \geq (\leq) R_2^{Pre}$ if and only if

$$\frac{\left(\overline{n}_{\hat{k}}^{Pre} - \overline{n}_{\hat{m}}^{Post}\right)}{\overline{n}_{\hat{m}}^{Post}} \geq (\leq) \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right),$$

as required.

Lastly, we prove (ii.b). If population distribution is asymmetric before and after the reorganisation then we (hypothetically) break down the proliferation into two steps as shown below:

$$\text{Asymmetric } \hat{k} \longrightarrow \text{Symmetric } \hat{m} \longrightarrow \text{Asymmetric } \hat{m}$$

In Step 1, the CP changes the number of AUs from $\hat{k}$ to $\hat{m}$ and also changes the population distribution from asymmetric to symmetric. In Step 2, the CP keeps the number of AUs constant at $\hat{m}$, but changes the population distribution from symmetric to asymmetric. From Theorem 5.2 (iii), we know that the rent accumulated in Stage 2 strictly decreases after the re-organization considered in Step 2.[20] If

$$\frac{\left(N_{\hat{k}}^{Pre}\hat{m} - N_{\hat{m}}^{Post}\hat{k}\right)}{N_{\hat{m}}^{Post}\hat{k}} \geq \hat{k}\left(\frac{1}{H_{\hat{k}}^{Pre}} - \frac{1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right),$$

then from (ii.a), we conclude that the rent accumulated in Stage 2 weakly decreases after the reorganisation considered in Step 1 and therefore, we conclude that the rent accumulated in Stage 2 across these steps strictly decreases after any reorganisation. ■

Proof of Theorem 5.6. From Theorem 3.3, we know that $R^{Pre} \geq (\leq) R^{Post}$ if and only if

$$\frac{\hat{m}\left(\tau_{\hat{m}}^{Post} - \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\omega_{\hat{m}}^{Post}\right)}{\hat{k}\left(\tau_{\hat{k}}^{Pre} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\omega_{\hat{k}}^{Pre}\right)} \geq (\leq) \frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}},$$

$\omega_{\hat{k}}^{Pre} = \dfrac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}$, $\omega_{\hat{m}}^{Post} = \dfrac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}$, $\tau_{\hat{k}}^{Pre} = \dfrac{\hat{k}}{H_{\hat{k}}^{Pre}}$, and $\tau_{\hat{m}}^{Post} = \dfrac{\hat{m}}{H_{\hat{m}}^{Post}}$. Substituting these expressions and cross-multiplying, we have $R^{Pre} \geq (\leq) R^{Post}$ if and only if

$$N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\frac{(\hat{m}-1)}{\hat{m}\sum_{i=i_m^*}^{m} s_{B_i}^2}\right) \geq (\leq) N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\frac{(\hat{k}-1)}{\hat{k}\sum_{i=i_k^*}^{k} s_{A_i}^2}\right).$$

[20] In Step 2, since $H_{\hat{m}}^{Pre} < 1$, we have $N_{\hat{m}}^{Post} = N_{\hat{m}}^{Pre} > H_{\hat{m}}^{Post}N_{\hat{m}}^{Post}$ implying that Theorem 5.2 (ii) applies.

Since $\sum_{i=i_k^*}^{k} s_{A_i}^2 = \frac{1}{\hat{k}} + \sum_{i=i_k^*}^{k} \Delta_{A_i}^2$ and $\sum_{i=i_m^*}^{m} s_{B_i}^2 = \frac{1}{\hat{m}} + \sum_{i=i_m^*}^{m} \Delta_{B_i}^2$, we have $R^{Pre} \geq (\leq) R^{Post}$ if and only if

$$N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}} - \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2}\right) \geq (\leq) N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}} - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\frac{(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right). \tag{20}$$

To prove (i), we consider two cases regarding the population distribution after the reorganisation: one where it is symmetric and another where it is asymmetric. First, assume that the population distribution after the reorganisation is symmetric. Since the population distribution is symmetric before and after the reorganisation, we have $H_{\hat{k}}^{Pre} = H_{\hat{m}}^{Post} = 1$ and $\sum_{i=i_k^*}^{k} \Delta_{A_i}^2 = \sum_{i=i_m^*}^{m} \Delta_{B_i}^2 = 0$. Next, observe that $N_{\hat{m}}^{Post} > \hat{m}(\hat{m}-1)$ implies that $N_{\hat{m}}^{Post} > \hat{m} + \hat{k} - 1$.[21] Rearranging and multiplying across by $(\hat{m} - \hat{k})$, we have

$$(N_{\hat{m}}^{Post}+1)(\hat{m}-\hat{k}) > (\hat{m}^2 - \hat{k}^2) \Leftrightarrow (\hat{m}^2 - \hat{k}^2) > \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\Big(\hat{m}(\hat{m}-1) - \hat{k}(\hat{k}-1)\Big),$$

and since $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$,

$$(\hat{m}^2 - \hat{k}^2) > \left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\hat{m}(\hat{m}-1) - \left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\hat{k}(\hat{k}-1).$$

[21] This is because $\hat{m}(\hat{m}-1) > \hat{m} + \hat{k} - 1$. To see this, observe that $\hat{m}(\hat{m}-1) - \hat{m} - \hat{k} + 1 = \hat{m}^2 - 2\hat{m} - \hat{k} + 1$. Since $\hat{k} \leq \hat{m} - 1$, $\hat{m}^2 - 2\hat{m} + 1 - \hat{k} \geq \hat{m}^2 - 2\hat{m} - \hat{m} + 2 = (\hat{m}-1)(\hat{m}-2) > 0$.

Therefore,

$$
\begin{aligned}
N_{\hat{k}}^{Pre}\hat{m}\left(\frac{\hat{m}}{H_{\hat{m}}^{Post}}-\left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\frac{(\hat{m}-1)}{1+\hat{m}\sum_{i=i_m^*}^{m}\Delta_{B_i}^2}\right) &= N_{\hat{k}}^{Pre}\left(\hat{m}^2-\left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\hat{m}(\hat{m}-1)\right)\\
&\geq N_{\hat{m}}^{Post}\left(\hat{m}^2-\left(\frac{N_{\hat{m}}^{Post}+1}{N_{\hat{m}}^{Post}}\right)\hat{m}(\hat{m}-1)\right)\\
&> N_{\hat{m}}^{Post}\left(\hat{k}^2-\left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\hat{k}(\hat{k}-1)\right)\\
&= N_{\hat{m}}^{Post}\hat{k}\left(\frac{\hat{k}}{H_{\hat{k}}^{Pre}}-\left(\frac{N_{\hat{k}}^{Pre}+1}{N_{\hat{k}}^{Pre}}\right)\frac{(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right),
\end{aligned}
$$

and using (20), we have $R^{Pre} > R^{Post}$ as required.

When assuming that the population distribution after the reorganisation is asymmetric, we (hypothetically) break down the proliferation into two steps as shown below:

$$
\text{Symmetric } \hat{k} \longrightarrow \text{Symmetric } \hat{m} \longrightarrow \text{Asymmetric } \hat{m}
$$

In Step 1, the CP increases the number of AUs from $\hat{k}$ to $\hat{m}$, but keeps the population distribution symmetric. In Step 2, the CP keeps the number of AUs constant at $\hat{m}$, but changes the population distribution from symmetric to asymmetric. We have already shown in the previous paragraph that the total rent accumulated declines in Step 1 and using Theorem 5.3 (ii), we know that the total rent further declines in Step 2, thereby implying that the total rent declines in this scenario as well.

Next, we prove (ii). Since the population distribution is asymmetric before and symmetric after the reorganisation, we have $H_{\hat{k}}^{Pre} < 1 = H_{\hat{m}}^{Post}$ and $\sum_{i=i_k^*}^{k}\Delta_{A_i}^2 > 0 = \sum_{i=i_m^*}^{m}\Delta_{B_i}^2$. Substituting these expressions in the LHS and RHS of (20) and since $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$, we have

$$
\begin{aligned}
&\left(N_{\hat{k}}^{Pre}\hat{m}^2 - N_{\hat{m}}^{Post}\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}\right) + \left(\left(N_{\hat{k}}^{Pre}+1\right)\frac{\hat{k}(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2} - \left(N_{\hat{m}}^{Post}+1\right)\hat{m}(\hat{m}-1)\right)\\
&\geq N_{\hat{k}}^{Pre}\left(\hat{m}^2 - \frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\frac{\hat{k}(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2} - \hat{m}(\hat{m}-1)\right)\\
&= N_{\hat{k}}^{Pre}\left(\hat{m}^2 - \frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\frac{\hat{k}(\hat{k}-1)}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2} - \hat{m}^2 + \hat{k}\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\hat{m}-\hat{k}\right)\\
&= N_{\hat{k}}^{Pre}\left(\hat{m}^2 - \frac{\hat{k}^2}{H_{\hat{k}}^{Pre}}\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\frac{\hat{k}^2\left(1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)}{\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)} - \hat{m}^2\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\hat{m}-\hat{k}\right)\\
&= \left(\frac{\hat{k}^2}{H_{\hat{k}}^{Pre}} - \hat{m}^2\right) + \left(N_{\hat{k}}^{Pre}+1\right)\hat{k}^2\left(\frac{\left(1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)}{\left(1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2\right)} - \frac{1}{H_{\hat{k}}^{Pre}}\right) + \left(N_{\hat{k}}^{Pre}+1\right)\left(\hat{m}-\hat{k}\right)\\
&> 0,
\end{aligned}
$$

whenever $\dfrac{(\hat{m}-\hat{k})}{\hat{k}} \geq \hat{k}\left(\dfrac{1}{H_{\hat{k}}^{Pre}} - \dfrac{1+\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}{1+\hat{k}\sum_{i=i_k^*}^{k}\Delta_{A_i}^2}\right)$ and $H_{\hat{k}}^{Pre} < \dfrac{\hat{k}^2}{\hat{m}^2}$, thereby implying that $R^{Pre} > R^{Post}$ as required.

When assuming that the population distribution after the reorganisation is asymmetric, we (hypothetically) break down the proliferation into two steps as shown below:

$$\text{Asymmetric } \hat{k} \longrightarrow \text{Symmetric } \hat{m} \longrightarrow \text{Asymmetric } \hat{m}$$

In Step 1, the CP increases the number of AUs and changes the nature of the population distribution from $\hat{k}$ to $\hat{m}$ and asymmetric to symmetric respectively. In Step 2, the CP keeps the number of AUs constant at

$\hat{m}$, but changes the population distribution from symmetric to asymmetric. We have already shown in the previous paragraph that the total rent accumulated declines in Step 1 and using Theorem 5.3 (ii), we know that the total rent further declines in Step 2, thereby implying that the total rent declines in this scenario as well. ■

B. Impact of Interior Nash Equilibrium Effort Levels on Fractionalisation

In this section, we formally establish Fact 4. We start by establishing two key lemmas.

Lemma B.1 establishes that when $N > (k-1)k$ then there exists a population distribution satisfying $n_1 = n_2 = \cdots = n_{k-1} \geq n_k$. It is worth noting that such a population distribution satisfies interior Nash equilibrium (INE) conditions because for all such population distributions, we have,

$$\sum_{i=1}^{k} n_i^2 = (k-1)n_1^2 + n_k^2 > (k-1)n_1^2,$$

as required.

Lemma B.1. *If $N > (k-1)k$ then there exists $d \geq 0$ and $1 \leq r \leq (k-1)$ such that $(k-1)(k+d)+r = N$.*[22]

Proof. We proceed by induction on N. For the base case, let $N = (k-1)k + 1$. Here, we can choose $d = 0$ and $r = 1$, satisfying $d \geq 0$ and $1 \leq r \leq k-1$. Thus, the base case holds. For the inductive hypothesis, assume that for $N = (k-1)k + l$ where $0 \leq l \leq L$ (for some positive integer L), there exist $d_l \geq 0$ and $1 \leq r_l \leq (k-1)$ such that $(k-1)(k+d_l) + r_l = N$. Now consider $N = (k-1)k + L + 1$. From the inductive hypothesis, we know that for $N = (k-1)k+L$, there exist $d_L \geq 0$ and $1 \leq r_L \leq (k-1)$ such that $(k-1)(k+d_L)+r_L = (k-1)k+L$. If $r_L < (k-1)$ then $N = (k-1)k+L+1 = (k-1)(k+d_L)+r_L+1$. Here, we can set $d = d_L \geq 0$ and $r = r_L+1$. Since $r_L < (k-1)$, $r = r_L+1 \leq (k-1)$ and $d \geq 0$ by the inductive hypothesis. On the other hand, if $r_L = (k-1)$. Then $N = (k-1)k+L+1 = (k-1)(k+h_L)+(k-1)+1 = (k-1)(k+d_L+1)+1$. Here, we can set $d = d_L + 1$ and $r = 1$. Clearly, $d \geq 0$ and $r = 1 \leq (k-1)$. Thus, the claim holds in this case as well. ■

Lemma B.2 implies that when comparing the sum of squares of population distributions that admit an INE we can, without loss of generality, compare population distributions that satisfy the property $n_1 = n_2 = \cdots = n_{k-1} \geq n_k$.

[22]This can be thought of as a direct consequence of the division algorithm as well.

Lemma B.2. *Consider N and k so that there exists some population distribution that admits an INE and let $(n_1, n_2, \ldots, n_k)$ one such population distribution such that there exists $1 < l < (k-1)$ with $n_1 = \cdots = n_l > n_{l+1}$. Then, the population distribution $(\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k)$ with $\hat{n}_i = n_1$ for $1 \le i < k$ and $\hat{n}_k = n_k + \sum_{i=l+1}^{k-1} (n_i - n_1)$ admits an INE and has higher sum of squares than the population distribution $(n_1, n_2, \ldots, n_k)$.*

Proof. By construction, it is obvious that $\sum_{i=1}^{k} \hat{n}_i = N$ and therefore, $(\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k)$ is indeed a population distribution.

Next, observe that

$$
\begin{aligned}
\sum_{i=1}^{k} \hat{n}_i^2 &= (k-1)n_1^2 + \left(n_k + \sum_{i=l+1}^{k-1} n_i - (k-l-1)n_1\right)^2 \\
&= (k-1)n_1^2 + n_k^2 + \sum_{i=l+1}^{k-1} n_i^2 + (k-l-1)^2 n_1^2 - 2(k-l-1)n_1 \sum_{i=l+1}^{k} n_i + \sum_{\substack{i,j=l+1 \\ i \neq j}}^{k} n_i n_j \\
&= l n_1^2 + \sum_{i=l+1}^{k-1} n_i^2 + n_k^2 + (k-l-1)n_1^2 + (k-l-1)^2 n_1^2 - 2(k-l-1)n_1 \sum_{i=l+1}^{k} n_i + \sum_{\substack{i,j=l+1 \\ i \neq j}}^{k} n_i n_j \\
&= \sum_{i=1}^{k} n_i^2 + (k-l-1)(k-l)n_1^2 - 2(k-l-1)n_1 \sum_{i=l+1}^{k} n_i + \sum_{\substack{i,j=l+1 \\ i \neq j}}^{k} n_i n_j \\
&= \sum_{i=1}^{k} n_i^2 + n_k^2 + \sum_{\substack{i,j=l+1 \\ i \neq j}}^{k} (n_i - n_1)\left(n_j - n_1\right) \\
&> \sum_{i=1}^{k} n_i^2 \\
&> (k-1)n_1^2,
\end{aligned}
$$

as required. ■

Now we are in a position to prove Fact 4. We only prove sufficiency as necessity follows directly from the formulation. Consider two population distributions $(\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k)$ and $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$ satisfying $\hat{n}_1 = \hat{n}_2 = \cdots = \hat{n}_{k-1}$ and $\tilde{n}_1 = \tilde{n}_2 = \cdots = \tilde{n}_{k-1}$ such that $\hat{n}_k < \tilde{n}_k$. We claim that $\sum_{i=1}^{k} \hat{n}_i^2 > \sum_{i=1}^{k} \tilde{n}_i$. To see this,

$$
\begin{aligned}
\sum_{i=1}^{k} \hat{n}_i^2 &= (k-1)\left(\tilde{n}_1 + \frac{(\tilde{n}_k - \hat{n}_k)}{(k-1)}\right)^2 + (\tilde{n}_k - (\tilde{n}_k - \hat{n}_k))^2 \\
&= (k-1)\tilde{n}_1^2 + \frac{(\tilde{n}_k - \hat{n}_k)^2}{(k-1)} + 2\tilde{n}_1(\tilde{n}_k - \hat{n}_k) + \tilde{n}_k^2 + (\tilde{n}_k - \hat{n}_k)^2 - 2\tilde{n}_k\,(\tilde{n}_k - \hat{n}_k) \\
&= (k-1)\tilde{n}_1^2 + \tilde{n}_k^2 + \frac{(\tilde{n}_k - \hat{n}_k)^2}{(k-1)} + 2\,(\tilde{n}_1 - \tilde{n}_k)\,(\tilde{n}_k - \hat{n}_k) + (\tilde{n}_k - \hat{n}_k)^2 \\
&> \sum_{i=1}^{k} \tilde{n}_i^2,
\end{aligned}
$$

as required.

Lastly, we present a lemma that characterizes the change in fractionalisation when an effectively expansive reorganisation transforms a symmetric population distribution of active AUs into an asymmetric one.

Lemma B.3. *When $\hat{m} > \hat{k} > 1$ and $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$, the following inequality holds:*

$$
\frac{(\hat{m}-1)\left(\dfrac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right)^2 + r_{\hat{m}}^2}{\hat{k}\left(\dfrac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2} < \frac{(\hat{m}-1)}{(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2 \tag{21}
$$

where $r_{\hat{m}}$ is the smallest positive integer such that $(\hat{m}-1)$ perfectly divides $N_{\hat{m}}^{Post} - r_{\hat{m}}$ and $\left(\dfrac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right) \geq r_{\hat{m}}$.

Proof of Lemma B.3. Since $\left(\dfrac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right) \geq r_{\hat{m}} > 0$, we have $r_{\hat{m}} \leq \dfrac{N_{\hat{m}}^{Post}}{\hat{m}} < \dfrac{N_{\hat{k}}^{Pre}}{\hat{k}}$ and combined with the fact that $(\hat{m}-1) > (\hat{k}-1)$, we have

$$
(\hat{k}-1)r_{\hat{m}}^2 - (\hat{m}-1)\left(\frac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2 < 0. \tag{22}
$$

Next, since $(N_{\hat{m}}^{Post} - r_{\hat{m}}) < N_{\hat{m}}^{Post} < N_{\hat{k}}^{Pre}$ and $(\hat{m}-1) \geq \hat{k}$, we have

$$
\left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right) \leq \left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{k}}\right) < \left(\frac{N_{\hat{m}}^{Post}}{\hat{k}}\right) < \frac{N_{\hat{k}}^{Pre}}{\hat{k}},
$$

and therefore,

$$\left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right)^2 - \left(\frac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2 < 0. \tag{23}$$

Combining (22), (23), and the fact that $(m-1) > (k-1) > 0$, we have

$$(\hat{m}-1)(\hat{k}-1)\left(\left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right)^2 - \left(\frac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2\right) + \left((\hat{k}-1)r_{\hat{m}}^2 - (\hat{m}-1)\left(\frac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2\right) < 0,$$

and since $N_{\hat{k}}^{Pre} \geq N_{\hat{m}}^{Post}$, we have

$$\frac{(\hat{m}-1)\left(\frac{N_{\hat{m}}^{Post} - r_{\hat{m}}}{\hat{m}-1}\right)^2 + r_{\hat{m}}^2}{\hat{k}\left(\frac{N_{\hat{k}}^{Pre}}{\hat{k}}\right)^2} < \frac{(\hat{m}-1)}{(\hat{k}-1)} \leq \frac{(\hat{m}-1)}{(\hat{k}-1)}\left(\frac{N_{\hat{k}}^{Pre}}{N_{\hat{m}}^{Post}}\right)^2,$$

as required. ■

Corollary B.1. *When $\hat{k} > \hat{m} > 1$ and $N_{\hat{m}}^{Post} \geq N_{\hat{k}}^{Pre}$, the following inequality holds:*

$$\frac{(\hat{k}-1)\left(\frac{N_{\hat{k}}^{Pre} - r_{\hat{k}}}{\hat{k}-1}\right)^2 + r_{\hat{k}}^2}{\hat{m}\left(\frac{N_{\hat{m}}^{Post}}{\hat{m}}\right)^2} < \frac{(\hat{k}-1)}{(\hat{m}-1)}\left(\frac{N_{\hat{m}}^{Post}}{N_{\hat{k}}^{Pre}}\right)^2 \tag{24}$$

where $r_{\hat{k}}$ is the smallest positive integer such that $(\hat{k}-1)$ perfectly divides $N_{\hat{k}}^{Pre} - r_{\hat{k}}$ and $\left(\frac{N_{\hat{k}}^{Pre} - r_{\hat{k}}}{\hat{k}-1}\right) \geq r_{\hat{k}}$.