Phase Transitions in Random Networks

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July 6 2019

Phase Transition

- Phase Transition: A small change in a local parameter results in an abrupt change in the global behaviour.
- On the Systemic Fragility of FinanceLed Growth, Metroeconomica, Vol. 66, Issue 1, pp. 158-186, 2015 by Amit Bhaduri, Srinivas Raghavendra and Vishwesha Guttal.

$$\frac{dP}{dt} = -P^3 - 5P^2 - 4P + D.$$

Example

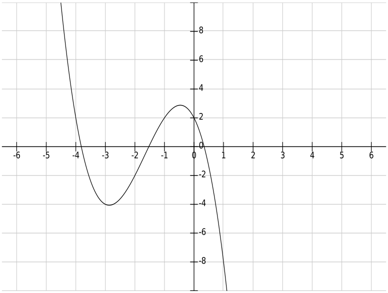


Figure: $f(x) = -x^3 - 5x^2 - 4x + D$ with $D = 2$

Example

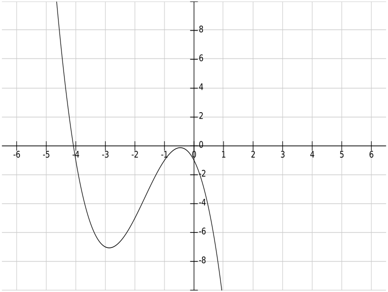
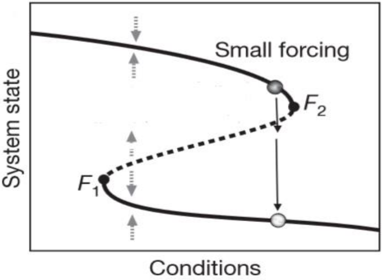


Figure: $f(x) = -x^3 - 5x^2 - 4x + D$ with $D = -1$

Bi-Stability and Hysteresis



Agent-based Models (Lux, 1995)

- Market has two types of traders.
- Fundamentalists who number N and buy (sell) one unit of stock when asset price is below (above) the fundamental price (p_F).
- Chartists or noise traders who also number N. Chartists are categorised either as optimists (N_O) or pesimists (N_P). $N_O + N_P = N$.
- Transitions: $O \to P$ at rate $\frac{N_P}{N}$ and $P \to O$ at rate $\frac{N_O}{N}$.

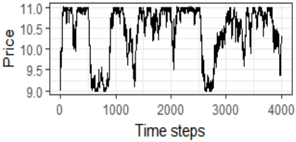
$$x := \frac{N_O - N_P}{N}.$$

- Price Dynamics

$$\frac{dP}{dt} = P\left(N_F(P_F - P) + N_C x\right).$$

- Equilibrium: $\frac{dP}{dt} = 0 \qquad \Rightarrow \qquad P = P_F + x.$

Price Realization



Herding

- Abrupt changes are purely due to random fluctuations. Not clear why this constitutes “herding”.
- What if transition rates have memory? Each transition of a particular type enhances the probability of the next transition being of the same type.

The Galton-Watson Branching Tree

- Start with one individual in generation zero: $Z_0 = 1$.
- Offspring Distribution: $\{p_k\}_{k\geq 0}$.
- X_{nj} = the number of children of the individual labeled j in generation n. The size of the population in generation $n+1$ satisfies

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{nj}; \qquad P(X_{nj} = k) = p_k.$$

- Mean Number of Offspring: $\mu = \sum_{k=1}^{\infty} kp_k$.

$$E[Z_{n+1}|Z_n] = Z_n\mu \qquad \Rightarrow \qquad E[Z_n] = \mu^n.$$

- $\frac{Z_n}{\mu^n}$ is a martingale.
- For $\mu \leq 1$ we have $Z_n \to 0$ almost surely.
- For $\mu > 1$ extinction probability η satisfies $\eta = \sum_{k=0}^{\infty} p_k\eta^k$.

Bond Percolation on the Random Grid

- Consider the infinite $d-$dimensional lattice $\mathbb{Z}^d$.
- Label each edge between neighboring sites open with probability p and closed with probability $1-p$ independent of everything else.
- Let C be the component containing the origin.
- Percolation Probability: $\theta(p) = P\left(|C| = \infty\right)$.
- The random grid is said to percolate if $\theta(p) > 0$.
- $\theta(p)$ is a monotonic function.
- Critical Intensity: $p_c := \inf\{p > 0 : \theta(p) > 0\}$.
- Non-trivial Phase Transition: $0 < p_c < 1$.

Non-trivial Phase Transition

- $p_c > 0$:

$$\begin{aligned}\theta(p) &\leq P(\text{There is an open path of length n from the origin}) \\ &\leq p^n(2d) \cdot (2d-1)^{n-1} \to 0,\end{aligned}$$

  as $n \to \infty$ for $p < (2d-1)^{-1}$.

- For any $d \geq 3$, $p_c(d) \leq p_c(2) = \frac{1}{2}$.

- For $p < p_c$ components sizes have exponentially decaying tails as do components not part of the infinite component in the case $p > p_c$.

Number of Infinite Components

- If the random grid percolates then there is w.p. 1 an infinite component.
  1. $P(\cup\{|C_x| = \infty\}) \leq \sum_x P(|C_x| = \infty)$ and $\theta(p) = P(|C_x| = \infty)$.
  2. $P(\text{There exists an infinite component}) \geq \theta(p) > 0$. Zero-One Law.

- There is at most one infinite component.
  1. Number of infinite components N is translation invariant. Ergodic Theory $\Rightarrow P(N = k) = 1$ for some $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$.
  2. Rule out $2 \leq k < \infty$.
  3. Burton-Keane argument.

Consequence of Uniqueness; Open Problem

- $p_c(2) = \frac{1}{2}$.
  1. $\theta(\frac{1}{2}) = 0$. Suppose Not… $\Rightarrow p_c \geq \frac{1}{2}$.
  2. Suppose $p_c > \frac{1}{2}$. Then $\theta(\frac{1}{2}) = 0 \Rightarrow P(\text{Left-right crossing of } B_n) \to 0$. $P(\text{Left-right crossing of } B_n) = P(\text{Top-down dual crossing of } B_n) = \frac{1}{2}$.

- SLE: Scaling limits of interface between open and closed islands at criticality.

- Open Problem: What happens at criticality for $d = 3$?
- Known for a long range percolation model.

The Erdös-Rényi Random Graph $G(n, p)$

- Graph with n vertices and an edge between any two pairs of vertices with probability p.
- Expected degree of a vertex $= (n-1)p$.

$$np \begin{cases} \to 0 & \text{Sparse Regime} \\ \to c \in (0,\infty) & \text{Thermodynamic Regime} \\ \sim c \log n & \text{Connectivity Regime} \\ \sim cn & \text{Dense Regime.} \end{cases}$$

The Thermodynamic regime: $p = \frac{c}{n}$

- Local Tree-like structure - no short cycles.
- Exploration Process: Start from some node r. Number of neighbors $\sim \text{Bin}((n-1), p)$. Need to discount for nodes already explored.
- If $c \leq 1$ the exploration process is dominated by a GW branching process and hence dies out w.p.1.
- If $c > 1$ the branching process approximation works well until component size stays below $n^{\frac{2}{3}}$.
- $\eta < 1 \Rightarrow$ component will grow to reach size $n^{\frac{2}{3}}$ with positive probability.
- Can't have two disjoint component of size larger than $n^{\frac{2}{3}}$. So only one giant component of size $O(n)$.
- Second largest component of size $O(\log n)$. Expected number of nodes not in the giant component is $\sim \eta n$

The Critical Window and Degree Distribution

- The Critical Window: $p = \frac{1}{n} + \beta n^{-\frac{1}{3}}$.
- Component sizes are $O(n^{\frac{2}{3}})$.
- Map components via the exploration process to the excursions of a reflected RW.
- Scaled excursions then converge to the excursions of a reflected BM with drift.

- Degree Distribution: If $p = \frac{c}{n}$ then $P(D = k) \approx \binom{n}{k} p^k (1-p)^{n-k} \rightarrow \frac{e^{-c} c^k}{k!}$, The Poisson distribution!

The Configuration Model

- Empirical data of real-world networks shows degree distribution to be heavy-tailed.
- Given a degree distribution F can we construct a graph on n vertices whose nodes have degree distribution F? Suppose the pmf of F is $\{p_k\}_{k\geq 0}$
- Let $d_1, d_2, \ldots, d_n \overset{i.i.d.}{\sim} F$.
- Endow vertex i with d_i half edges.
- Randomly pair these half edges. Ignore self loops and multiple edges.

Size-biasing and Phase Transition

- Start exploring from some vertex o. This vertex has degree k with probability p_k. Not true for the subsequent vertices in the exploration branching process.
- Size Biasing: A first generation vertex with degree m is m times as likely to be chosen to be the progeny of o as a vertex with degree 1!
- The offspring distribution in the first and subsequent generation then becomes

$$q_{k-1} = \frac{kp_k}{\mu} \qquad \text{where} \qquad \mu = \sum kp_k.$$

- Effective Mean Number of Progeny: $\nu = \frac{1}{\mu} \sum k(k-1)p_k$.
- Phase Transition at $\nu = 1$.
- Open Problem: What is the size of the largest component when $\nu < 1$? Conjecture: $O(n^{\frac{1}{(\gamma-1)}})$ when $p_k \sim Ck^{-\gamma}$, $\gamma > 3$.

Power Laws, Preferential Attachment

- Dynamic process to obtain a graph with heavy tailed degree distributions.
- Start at $t = 0$ with two vertices with an edge between them.
- At each time t add a vertex and connect it to one vertex in the existing graph. A vertex of degree k is chosen with probability proportional to $f(k)$.
- If $f(k) = k^\alpha$, $\alpha < 1$, then $p_k \approx ck^{-\alpha}e^{-ck^{1-\alpha}}$.
- If $f(k) = k^\alpha$, $\alpha > 1$, then there is one vertex with degree $O(t)$ and all other vertices have degree $O(1)$.
- If $f(k) = a + k$, and $a > -1$ then $p_k \sim Ck^{-(3+a)}$.

The Poisson Random Connection Model

- The RCM is a random graph.
- Vertex set is a Poisson Point Process $\mathcal{P}_\lambda$ of intensity $\lambda > 0$ in $\mathbb{R}^d$, $d \in \mathbb{N}$.
  1. For $A \subset \mathbb{R}^d$, the number of points in A follows a Poisson distribution with mean $\lambda|A|$:

$$P(\text{The number of points in } A = k) = e^{-\lambda|A|}\frac{(\lambda|A|)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

  2. For $A, B \subset \mathbb{R}^d$ the number of points in A and B are independent.
- Connection Function: $g : \mathbb{R}^d \to [0, 1]$.
- Edges: There is an edge between $x, y \in \mathcal{P}_\lambda$ with probability $g(x - y)$ independently of everything else. Dependency!!!

Applications

- Physics: Bonds in Particle Systems
- Epidemiology: Infected herd at location x infecting another at location y.
- Telecommunication: Communication between two transmitters.
- Biology: Sensing between two cells.

Related Models

- Random Geometric Graph: $g(x) = 1$ if $|x| \leq 2r$ and zero otherwise.
- SINR Graph: Edge between x_i, x_j if the signal-to-noise-plus-interference-ratio

$$SINR = \frac{PE_{ij}\ell(x_i, x_j)}{N + \gamma \sum_{k \neq i,j} PE_{kj}\ell(x_k, x_j)} > T.$$

- Inhomogenous RCM:

$$g(x) = 1 - \exp\left(-\eta \frac{W_x W_y}{|x|^\alpha}\right).$$

Independent Weights. $\qquad P(W_x > w) = w^{-\beta}, \qquad w \geq 1.$

- RCM with preferential attachment: (Jacob and Morters (2015))

Phase Transition

- Phase Transition: A small change in a local parameter results in an abrupt change in the global behaviour.
- The phase transition of interest is the size of the largest component: multi-hop transmission, conductivity, spread of epidemics
- Infinite System: Transition from finite connected components to an infinite component.
- Finite System: Components logarithmic in size to a giant component which covers a non-trivial fraction of the nodes.

Percolation

- Assume there is a point at the origin O. Let C be the component containing the origin.

- Percolation Probability: $\theta(\lambda) = P(|C| = \infty)$.
- The RCM is said to percolate if $\theta(\lambda) > 0$.

- If the RCM percolates then there is w.p. 1 a unique infinite component.

- Critical Intensity: $\lambda_c := \inf\{\lambda > 0 : \theta(\lambda) > 0\}$.
- $\theta(\lambda)$ is a monotonic function.
- Non-trivial Phase Transition: $0 < \lambda_c < \infty$.

Non-trivial Phase Transition in the RCM

- The points connected to O form an inhomogenous Poisson point process of intensity $\lambda g(x)$.
- Expected Degree $= \lambda \int_{\mathbb{R}^d} g(x)dx$.
- No non-trivial phase transition if $\int_{\mathbb{R}^d} g(x)dx$ equals zero or ∞.

Theorem (Penrose 1991)

Suppose $0 < \int_{\mathbb{R}^d} g(x)dx < \infty$. *Then there exists a* $\lambda_c \in (0, \infty)$ *such that* $\theta(\lambda) = 0$ *for* $\lambda < \lambda_c$ *and* $\theta(\lambda) > 0$ *for* $\lambda > \lambda_c$.