The zero coupon yield curve

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Product definitions

Coupon versus zero-coupon bonds

- In the olden days, bonds came with attached "coupons".
- You had to cut off the coupon and mail it in, to get interest.
- A bond that had no coupons, but only paid back Rs.100 at expiration, was a "zero coupon bond".

Cashflows

- A hypothetical two-year 12% coupon bond, and a hypothetical two-year zero coupon bond:

t	0.500	1.000	1.500	2.000
Coupon bond:	6	6	6	106
Zero coupon bond:				100

- We use the notation $(c_1, t_1), (c_2, t_2), \ldots, (c_N, t_N)$ to denote a bond with N cashflows.

Strips

- You buy this coupon bond:

t	0.500	1.000	1.500	2.000
Coupon bond:	6	6	6	106

- You issue four zero-coupon bonds out of the four cashflows.
- Legal engineering so that your bankruptcy doesn't affect the buyers of your zero coupon bonds.
- A way of allowing RBI to continue to issue coupon bonds, but trading can then be done in zero coupon bonds.

Key intuition

- A coupon bond is a *portfolio* of zero coupon bonds.

Elementary compound interest

Compound interest

$$\mathrm{PV} = \frac{C}{(1+r)^T}$$

r is a feature of the economy.

Continuous compounding

- In continuous time, it works out that:

$$\text{PV} = e^{-rT} C$$

- Very convenient, for things like:

$$\int_0^\infty e^{-rt} c(t) dt$$

- Note : if $s = \log(1 + r)$ then $e^s = (1 + r)$.

A simple world - one r, one formula

Suppose there are cashflows $(c_1, t_1), \ldots, (c_N, t_N)$. The economy tells us r, and then:

$$\text{PV} = \sum_{i=1}^{N} \frac{c_i}{(1+r)^{t_i}}$$

The yield curve

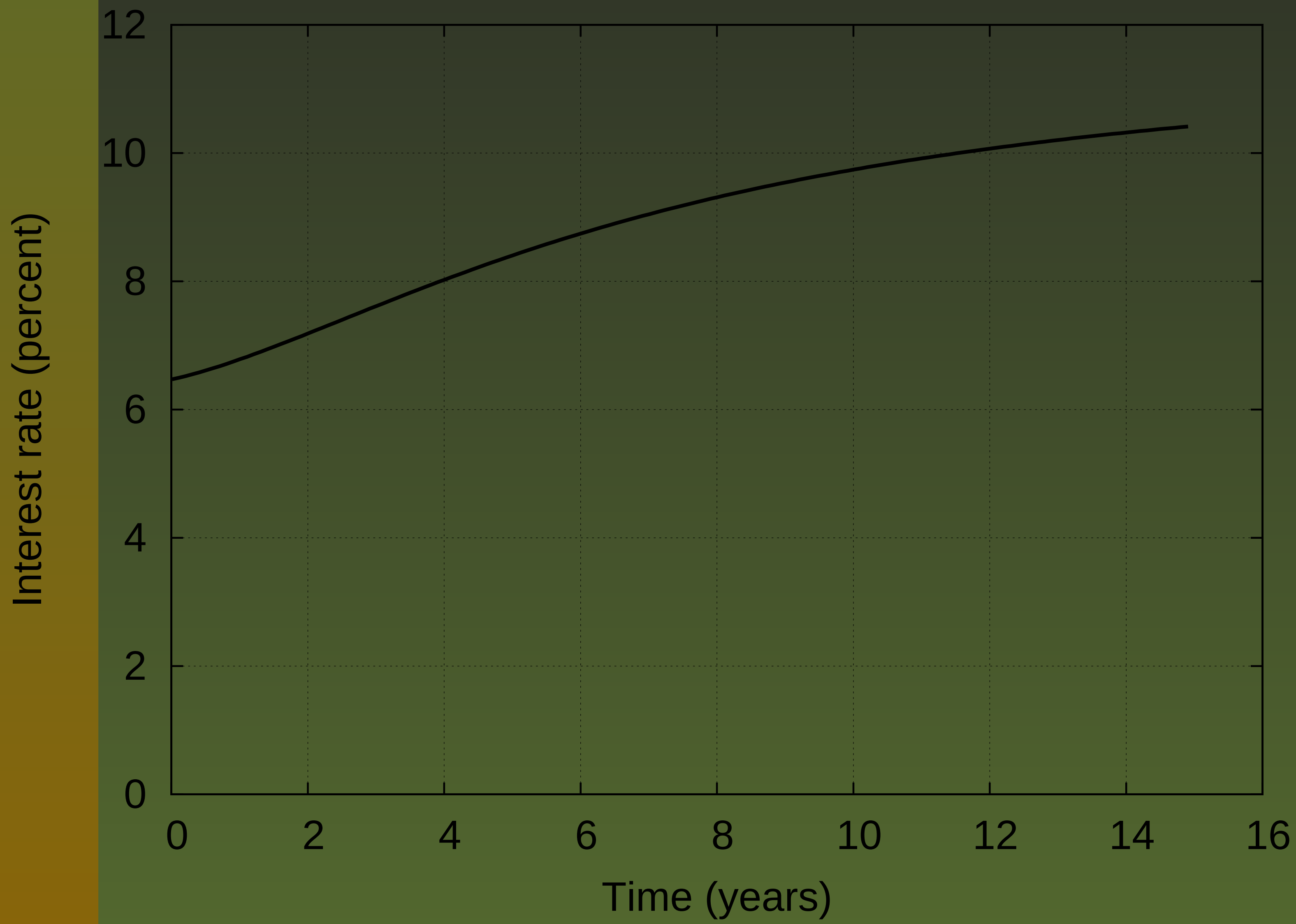
Unfortunately,

- We have an entire *yield curve*,
- with different r for different t.
- We can't discount all future cashflows at the same rate.

$$\mathrm{PV} = \frac{C}{(1 + z(T))^T}$$

The economy tells us all interest rates, i.e. the function $z(T)$.

Example – 2 August 2001



Pricing a set of cashflows

If there are cashflows $(c_1, t_1), \ldots, (c_N, t_N)$, the economy tells us $z(t)$, and then:

$$\mathrm{PV} = \sum_{i=1}^{N} \frac{c_i}{(1 + z(t_i))^{t_i}}$$

Jargon

- “The yield curve”
- “Spot yield curve”
- “Zero coupon yield curve”
- “Zero curve”

are all the same.

Example – a zero coupon bond on 2 August 2001

- A bond pays Rs.100 on 2 August 2003.

Example – a zero coupon bond on 2 August 2001

- A bond pays Rs.100 on 2 August 2003.
- For $t = 2$, $z(t) = 7.19\%$, so

$$\begin{aligned} \text{PV} &= \frac{100}{(1.0719)^2} \\ &= \text{Rs.87.03.} \end{aligned}$$

Using the yield curve in making NPVs of cashflows

Example - a coupon bearing bond on 2 August 2001

- A bond has cashflows:

0.500	1.000	1.500	2.000
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6.611	6.786	6.982	7.190

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- Make the NPV of each cashflow:

5.811	5.619	5.422	92.257

Sums to Rs.109.11.

Intuition

- Case 1: You have 20 of Bond 1 and 20 of Bond 2
  1. Bond 1 pays Rs.100 in 1 year.
  2. Bond 2 pays Rs.100 in 2 years.
- Case 2: You have 20 of Bond 3
  1. Bond 3 has one cashflow of Rs.100 in 1 year, and one cashflow of Rs.100 in 2 years.
- *Case 1 and 2 are identical.*

Intuition

- A bond portfolio is a portfolio of coupon / zero-coupon bonds.
- Every coupon bond is a portfolio of zero coupon bonds.
- Once you know how to price a zero coupon bond, you can price every portfolio.

so we use the steps:

- Break a portfolio into a list of bonds.
- Break each bond into a list of cashflows.
- Make a comprehensive list of all cashflows.
- PV each cashflow using the correct discount rate.

Data issues

NSE ZCYC database

- NSE ZCYC database : from 1/1/1997 onwards.
- India's first term structure database.
- Based on value weighted averages from WDM.
- Side effect: generates "model price" for every bond for every day.

Data problems

- Hierarchy of sharpness:
  1. Call auction
  2. (bid+offer)/2 in realtime
  3. 30 minute average for closing price
  4. Full-day VWA from order book market.
  5. Full-day VWA from OTC market.
- Don’t trust WDM VWA like you trust (say) the closing price of Reliance.

Market price versus model price

- The market's VWA has vagaries from bond to bond, from day to day.
- The model price is pure NPV, averaging out pricing vagaries across bonds.
- Should be used in regulation of valuation by SEBI, RBI, etc.

Making the information sharper

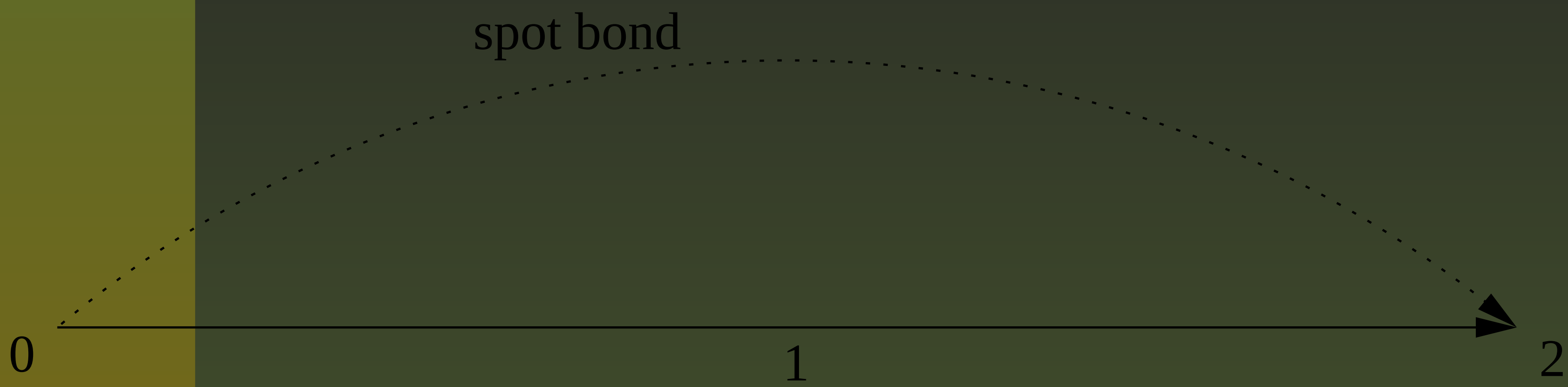
- Ideally, it should be possible to take (bid+offer)/2 for all GOI bonds, intra-day, and make a curve.
- Every time bid or offer changes, the curve would be recomputed.
- Analogy: You can compute a volatility smile for Nifty options in realtime.
- Bottleneck: non-transparency of the spot market for GOI bonds.

Forward rates

Notion of forward rate

- A spot bond is lending Rs.100 on date 0 and getting repaid on date t_2.
- Forward contract: agree on date 0, to lend Rs.100 on date t_1 and get repaid on date t_2.
- As with all forward contracts: no optionality, locked-in contract.
- The interest rate $r_{t_1}^{t_2}$ is called "a forward interest rate".
- The spot yield curve directly shows all the r_0^t values.

The zero curve embeds all forward rates!



- Case 1: buy a bond for Rs.100, yields $100(1 + r_0^2)^2$ after two years.

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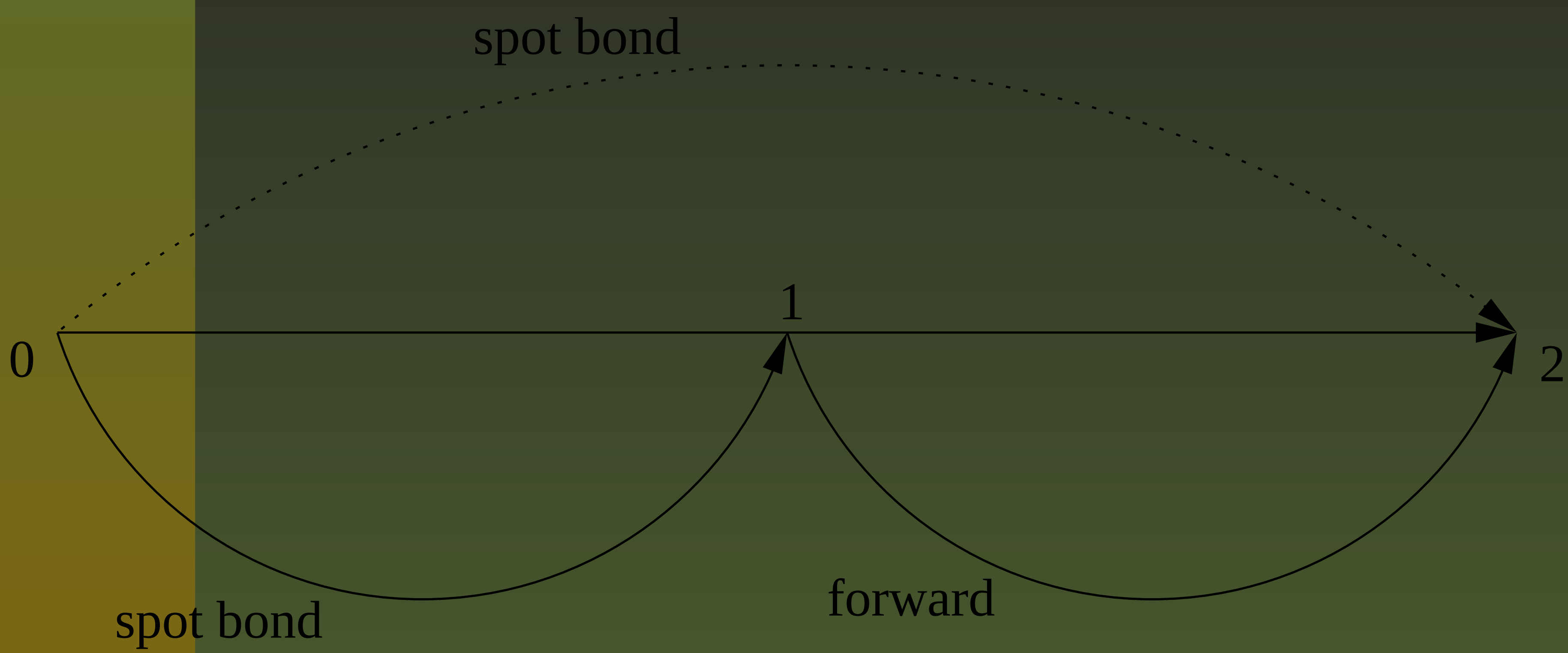

- Case 1: buy a bond for Rs.100, yields $100(1+r_0^2)^2$ after two years.
- Case 2: buy a one-year bond and enter into a following one-year forward contract.

The zero curve embeds all forward rates!

- Case 1: buy a bond for Rs.100, yields $100(1+r_0^2)^2$ after two years.
- Case 2: buy a one-year bond and enter into a following one-year forward contract.

$$
\begin{aligned}
100(1+r_0^2)^2 &= 100(1+r_0^1)(1+r_1^2) \\
(1+r_1^2) &= \frac{(1+r_0^2)^2}{(1+r_0^1)}
\end{aligned}
$$

An arbitrage relationship

$$(1 + r_1^2) = \frac{(1 + r_0^2)^2}{(1 + r_0^1)}$$

- If this is violated, it is an arbitrage opportunity.
- Given enough arbitrage capital and brains in the system, this will never be violated.

Key insights

- There is only one pricing kernel in the economy: the zero coupon yield curve.
- If you know the zero coupon yield curve, you know all forward rates.
- Interest rates read off the zero coupon yield curve pertain to *spot* interest rates (i.e. $t_1 = 0$). Hence the vocabulary “spot yield curve”.

Par yield curve

Par bond

- Suppose we decide to issue a 10-year bond today, with six monthly coupons.
- What should the coupon be, so that the NPV of the bond works out to Rs.100?
- “Par bond” : Rs.100 on date of issue.

“Par yield curve”

- Imagine a graph where T is on the x axis, and the coupon rate on the par bond is on the y axis.
- This is called the “par yield curve”.

No bugs here

- There are no logical flaws in the concept of the par bond or in the concept of the par yield curve.

YTM

The notion of YTM

- We know that the bond price comes out of:

$$p = \sum_{i=1}^{N} \frac{c_i}{(1 + z(t_i))^{t_i}}$$

- Suppose we want to insist there is only one interest rate $\bar{r}$ and look at the formula:

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- Fact: For every p, there is a $\bar{r}$, called “YTM”, so that the equation is satisfied.

Understanding YTM

- Every bond price can be remapped into a YTM, and vice versa.
- Each YTM is based on thinking the yield curve is flat.
- The YTM of one bond cannot be used for pricing another bond, except if the yield curve is flat, in which case all bonds have the same YTM.

What is wrong with YTM

- If I give you a new bond, you cannot use YTMs of existing bonds to price it.
- YTM is not a pricing kernel.
- YTM believes the yield curve is flat; it is not a path to thinking and understanding a non-flat yield curve.
- It plays no role in the economics of fixed income markets.

Myths

“You can only have a zero coupon yield curve if the country trades lots of zero coupon products”.

It is perfectly feasible to estimate the zero coupon yield curve off market prices for coupon bonds.

"I think in YTM, so a zero coupon yield curve is not useful to me".

You can make the NPV of a set of cashflows using the zero coupon yield curve, and convert that into a YTM.

“I have been in the fixed income market for 20 years and have never needed the zero coupon yield curve.”

- Interest rates were fixed till recently.
- Looking forward, complexity is escalating, so you’ll need theory. The theory is all based on the zero coupon yield curve.

“The zero coupon yield curve can only work when most market participants use it.”

- Jupiter does not need to use Newton’s laws, for Newton’s laws to work.
- Black/Scholes worked well before anyone on the options market knew it.
- “Technical analysis” doesn’t work, even though many market participants use it.
- Salomon Brothers in the late 1970s, early 1980s.

Metadata

Also see

Web page on Indian fixed income:
http://www.mayin.org/~ajayshah/FIXEDINCOME/index.html